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A CORRELATION OF RING-SHAPED CHROMOSOMES WITH VARIEGATION IN ZEA MAYS

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It has been concluded from cytological and genetical investigations that ordinary rod-shaped chromosomes perpetuate themselves through nuclear cycles by splitting into two similar halves, which pass freely to opposite poles of a mitotic spindle figure. Through such behavior the same genetic constitution in every nucleus of the plant is maintained. The evidence presented here indicates that the size and genetic constitution of ring-shaped chromosomes are not the same in all the nuclei of the plant. In one cell or a group of cells it may be larger or smaller than the original ring chromosome contributed by one of the gametes. The method by which a ring-shaped chromosome becomes reduced in size by loss of a section, or even totally eliminated from the cell, has not been determined. Nevertheless, somatic elimination of a section of a ring-shaped chromosome or a loss of the ring altogether should result in variegation if a genetic marker is present in the ring. If a normal chromosome carrying a recessive gene has as its homologue a ring-shaped chromosome carrying the dominant allelomorph, sporadic loss during development of the ring chromosome or a section of it which carries the dominant will result in groups of cells possessing only the recessive gene.

Eight cases of ring-shaped chromosomes have been studied. In all of these a diminution in size or loss of the ring chromosome was observed. One case was found in untreated material. The other cases arose in the progeny of x-rayed pollen. In five of these instances genetic markers were present in the chromosomes concerned.

All the variegated plants described in this paper appeared in the cultures of Dr. L. J. Stadler. The author is indebted to Dr. Stadler for the privilege of examining these plants for the presence of ring-shaped chromosomes.

The first case to be described involved the gene *B*, which is associated with sun-red plant color. The normal *B-lg* chromosome is easily dis-

tinguishable from the others of the complement by its relative size, the position of the insertion region and the position and size of its knob. The regional location of the gene liguleless (*lg*) and the probable locations of the genes *B* and virescent seedling (*v*$_4$) on this chromosome are given in figure 1.***

Pollen containing the genes *Lg*, *B* and *V*$_4$ was x-rayed and then placed upon silks of untreated plants possessing the recessive allelomorphs, *lg*, *b* and *v*$_4$. The progeny included a seedling which was classified as liguleless (*lg*) and probably virescent (*v*$_4$). As the plant developed, stripes of red and green tissue were noted on the stalks and leaf sheaths. The red tissue indicated the presence of the dominant gene *B;* the green tissue indicated the loss of this gene during development. Examination of the mid-prophase of meiosis showed, in many cells from different anthers, a large ring-shaped chromosome synapsed with the normal *b-lg* chromosome. The ring was deficient to the extent indicated by the regions beyond the arrows in figure 1. The breaks in the normal chromosome to produce this ring probably occurred a short distance from the end of the

FIGURE 1

short arm, thus removing the *Lg* gene, and through the knob on the long arm possibly removing the *V*$_4$ gene. In many sporocytes, however, the size of the ring was obviously diminished, this frequently involving a loss of the remaining portion of the knob. Very few sporocytes showed a total loss of the ring chromosome.

This evidence strongly suggests that the loss of the *B* gene, indicated by the green regions in the stalk and leaf sheaths, is due to the absence of that section of the ring chromosome which carries the *B* gene.

A second case of variegation involving the gene *B* was found to be associated with a ring-shaped *B-lg* chromosome. In this case, *lg* and *v*$_4$ were not marked. Here, also, diminished rings were observed.

A third case of variegation involving the *Pl* gene associated with purple plant color. X-rayed pollen from a plant homozygous for the dominant gene *Pl* was placed upon silks of a plant carrying only the recessive allelomorph, *pl*. The variegation appeared as stripes of purple and sun-red plant color. The locus of the *Pl* gene is toward the middle of the long arm of the satellited chromosome. In the sporocytes of this plant there were 9 normal appearing bivalents, a single satellited chromosome and a small ring-shaped chromosome. Anthers from different regions of the tassel differed in the size of the rings predominating in their

sporocytes. In each anther there were patches of related cells with rings much smaller than the predominating type for the anther. Moreover, the chromosome affected is the one known to carry the gene which is involved in the variegation.

The other two cases examined involved the brown mid-rib gene (bm_1) which is associated with brown color in the veins of the leaves and leaf sheaths. This effect is due to a brown coloration in the walls of the cells. Pollen from Bm_1 plants was x-rayed and placed upon silks of bm_1 plants. Variegation expressed itself as patches of brown and of colorless cell walls visible in cross-sections of the stem, or as regions with brown veins on the stalk and leaf.

In the first of these two cases metaphase and anaphase *I* figures showed 10 bivalents plus a fragment chromosome, a total of 21 chromosomes. Examination of mid-prophase of meiosis revealed the fragment as a small ring-shaped chromosome. Furthermore, a deletion in the short arm of the bm_1 chromosome comparable in extent to the most frequently observed size of the ring chromosome was also present. There was one extra insertion region to be accounted for. It is difficult to avoid the conclusions that the insertion region was divided in the formation of the ring and the deleted rod and that both sections of the insertion region were capable of functioning. The ring chromosome did not synapse with the homologous region in the normal chromosome (see McClintock, 1932).[1] The position of the buckling in the normal rod chromosome which compensates for the deficiency in the deleted rod chromosome usually appeared at a little distance from the insertion region in the short arm of the bm_1 chromosome. Evidence recently obtained makes it difficult to interpret these figures with regard to homologous association and thus to be sure of the location of the deficiency. This being so, no decisive statement can be made at this time regarding the origin of the ring and the division of the insertion region.

Diminution in size as well as loss of the ring-shaped chromosome occurred in this plant. Therefore, it is likely that the Bm_1 gene was included in the ring chromosome at the time of its formation from the normal rod chromosome and that the diminution in size or loss of the ring chromosome was responsible for the variegation.

The second case of variegation involving the brown mid-rib gene was similar in some aspects to the one described above. At metaphase and anaphase *I* there were 10 bivalents plus a tiny fragment chromosome, or 21 chromosomes in all. In the mid-prophase of meiosis the fragment chromosome was seen to be a very small ring composed of not more than several chromomeres. In synapsis of the normal bm_1 chromosome derived from the female with the Bm_1 chromosome derived from the male there was no indication of buckling in the normal chromosome to compensate

for a deletion. Since the ring representing the deleted portion was so small, it is not improbable that the deficient chromosome was slightly stretched to compensate for the small loss. Pollen examination indicated that the plant was approximately 50 per cent sterile. It could be concluded, therefore, that a deficiency was present in one of the rod chromosomes. A deficiency in the Bm_1 chromosome corresponding to the ring fragment would account for this sterility in view of the fact that complete absence of the ring fragment was noted in groups of sporocytes or even large sections of the tassel. Furthermore, anaphase *I* was characterized by the frequent loss of the ring fragment in those sporocytes possessing it. The evidence, taken as a whole, suggests that in this, as in the other Bm_1 case, a deletion process involving the Bm_1 region of the chromosome bisected the insertion region producing a ring fragment and a deficient rod chromosome each with a functional insertion region.

Large sections of an anther or even whole branches of the tassel were found to lack the ring-shaped chromosome. It is probable, therefore, that variegation in this case was more commonly associated with complete elimination of the ring chromosome than with diminution in size.

Of a total of 9 variegated plants which were obtained from x-ray treatment, 6 were examined cytologically. One of these gave such poor cytological figures that no definite conclusions could be drawn; in the small amount of material available, no evidence of a ring chromosome was visible. It is possible that the material examined represented a region in which the ring had been lost. It is also possible that some other chromosomal condition was responsible for the variegation, but the material was too poor to yield suggestive evidence on this point. Four of the other five plants showed a ring-shaped chromosome which was obviously derived from the chromosome known to be associated with the gene involved in the variegation. The fifth case, the second $Bm_1 - bm_1$ variegation, possessed a tiny ring-shaped chromosome but cytological evidence of its origin from the Bm_1 chromosome was not available. The presence of the ring chromosome in some cells and its absence in others, coupled with the appearance of variegation, indicated its origin as being similar to that in the other four cases.

Decrease in size of the ring-shaped chromosome through loss of constituent sections was strikingly manifest in all of the 8 cases of such chromosomes examined. In the ring with a knob marker, the decrease was frequently associated with a loss of the knob. The extent of loss ranged from a relatively small to a very large section of the chromosome. Complete elimination of the ring was observed also, this being especially frequent in those cases having the smaller rings. The evidence from the cases with genetic markers in the affected chromosomes suggests that these losses are responsible for the variegation observed in the plants.

Two other points not directly connected with variegation are of interest with regard to the behavior of ring-shaped chromosomes. These rings may increase in size and in number during the course of somatic divisions. Evidence for increase in the size of the ring was obtained not only in measurements showing such increases in cases where a more or less accurate estimate of the size of the original ring could be made, but also in the duplication of a visible marker within the ring. The ring chromosome in the first case described possessed a fragment of a knob. If the ring maintained itself unchanged throughout successive nuclear divisions, only one knob should be present regularly. In very rare cases 2 or even 4 knobs have been observed in the same ring chromosome. Here, again, where such duplication is evident in the rings, related cells in a group show similar conditions. Occasional accumulation of two or more rings in the same cell was observed in isolated cells or in groups of related cells. The several rings may or may not be of like size.

Investigations which should lead to an understanding of the mechanism responsible for the frequent decrease and occasional increase in size of the rings or for their loss have not as yet been conducted. Lack of uniformity in the splitting plane could give rise to a double sized ring with two insertion regions or cause split halves of the ring to become interlocked. Subsequent movement of the two insertion regions toward opposite poles at anaphase would cause breaks in the ring chromosomes and thus produce changes in their size and constitution. The figures of two interlocking rings in a somatic metaphase in *Crepis* given by Nawashin[2] are suggestive. It is noteworthy that in the decrease or increase in size of the rings in *Zea* nothing but rings have been observed to come from rings.

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*** A deficiency involving V_4 showed this gene to be in the long arm of the *B-lg* chromosome. Its assignment to a locus between the knob and the end of the chromosome is based only on the case described here. This plant was classified as a probable v_4. More evidence is necessary before a definite locus within the long arm can be given.

[1] McClintock, B., *Proc. Sixth Internat. Cong. Genet.*, **2**, 126–128 (1932).

[2] Nawashin, M., *Univ. Calif. Pub. Agr. Sci.*, **6**, 95–106 (1930).

CRITICAL TORSIONAL OSCILLATIONS OF A ROTATING ACCELERATED SHAFT

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We assume that the forces applied to the shaft have a variable part which is a moment of constant amplitude $M(x)$ per unit length, distributed along the shaft, and varying with a frequency proportional to the angular velocity.

From the solution corresponding to the harmonic steady state vibration, we deduce, by using Heaviside's expansion, the motion due to a sudden application of the moment $M(x)$. This enables us to compute the effect of the moment when applied with a linearly increasing frequency.

In this analysis the damping will be neglected. In case of a viscous damping the linear character of the equations is not affected and the same method might be used. We did not carry this calculation for two reasons:

1. The exact result will be in general complicated, and involve viscous friction coefficients which will not be very accurately known. Besides, the effect of friction might be roughly taken into account by considering the steady state solution.

2. In most cases the damping is not viscous but due to the hysteresis or internal friction of the material. This is proved by the experimental fact that the energy absorbed in the vibration of elastic bodies is proportional to the frequency and not to its square. This effect might be taken roughly into account by energetic considerations.

The equation of vibration of the shaft may be written,

$$M(x, t) + \frac{\partial}{\partial x}\left[K(x)\frac{\partial \theta}{\partial x}\right] = I(x)\frac{\partial^2 \theta}{\partial t^2}, \qquad (1)$$

where θ is the angular coördinate of a cross-section, $K(x)$ the torsional rigidity and $I(x)$ the moment of inertia per unit length.

In order to find a solution of this equation, we will first study the behavior of the free oscillations given by the equation,

$$\frac{\partial}{\partial x}\left[K(x)\frac{\partial \theta}{\partial x}\right] = I(x)\frac{\partial^2 \theta}{\partial t^2}.$$

This equation has an infinite number of solutions of the type $\theta = \Theta_i(x)e^{i\omega_i t}$ corresponding to the free harmonic oscillations of the shaft with the given boundary conditions.

Each of the functions $\Theta_i(x)$ is a characteristic function which satisfies the Sturm-Liouville differential equation,

$$\frac{\bar{d}}{dx}\left[K(x)\frac{d}{dx}\,\Theta_i(x)\right] + \omega_i^2 I(x)\Theta_i(x) = 0 \qquad (2)$$

and also the given boundary conditions. These functions are related by the orthogonality condition,

$$\int_0^a \Theta_m(x)\,\Theta_n(x)\,\bar{d}x = 0 \quad m \neq n. \qquad (3)$$

This analysis is valid when the inertia moments are concentrated instead of distributed, for we can always approximate concentrated loads by a continuous distribution.

We now assume that the applied moment is harmonic of the form, $M(x, t) = M_0(x)e^{i\omega t}$. The solution of equation (1) may then be written

$$\theta(x, t) = \theta_0(x)e^{i\omega t},$$

and the equation becomes an ordinary differential equation,

$$M_0(x) + \frac{\bar{d}}{dx}\left[K(x)\frac{\bar{d}\theta_0}{\bar{d}x}\right] + \omega^2 I(x)\,\theta_0 = 0. \qquad (4)$$

For our purpose the best way of solving this equation is to expand $\theta_0(x)$ in a series of orthogonal functions $\Theta_i(x)$,

$$\theta_0(x) = \sum_{i=1}^{\infty} A_i\Theta_i(x).$$

Substituting this expression into equation (4) and taking into account the identity (2), we find,

$$M_0(x) = I(x)\sum_{i=1}^{\infty} A_i(\omega_i^2 - \omega^2)\Theta_i(x).$$

If we multiply both sides by $\Theta_k(x)$ and integrate along the shaft between the limits (o, a), due to the orthogonality condition (3), all the unknown constants A_i are eliminated except A_k; we finally get

$$A_i = \frac{C_i}{\omega_i^2 - \omega^2}, \qquad C_i = \frac{\int_0^a M_0(x)\,\Theta_i(x)\bar{d}x}{\int_0^a I(x)\,\Theta_i^2(x)\bar{d}x}.$$

The required solution for the forced harmonic vibration is

$$\theta_h(x, t) = e^{i\omega t}\sum_{i=1}^{\infty}\frac{C_i\Theta_i(x)}{\omega_i^2 - \omega^2}. \qquad (5)$$

We will now calculate the motion of the shaft due to a sudden applied

distributed moment $M_0(x)$. Consider the following well-known integral taken in the complex plane

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega}\,d\omega = \begin{cases}1 & t>0\\ 0 & t<0\end{cases}.$$

The path of integration is supposed to pass below the real axis. The applied moment may hence be represented as a sum of harmonic forces with a continuous set of frequencies by the expression

$$\frac{M_0(x)}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega}\,d\omega.$$

This integral can be expressed as a contour integral, if we note that for $t>0$ we may add to the path of integration the infinite half circle passing through $(-\infty, +i\infty, +\infty)$. Hence for $t>0$

$$\frac{M_0(x)}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega}\,d\omega = \frac{M_0(x)}{2\pi i}\oint\frac{e^{i\omega t}}{\omega}\,d\omega.$$

To each of the harmonic moments composing this expression corresponds a motion given by equation (5); the total motion due to the sudden moment will be the sum of all these elementary motions and expressed by

$$\theta_2(x,t) = \sum_{i=1}^{\infty}\frac{C_i\Theta_i(x)}{2\pi i}\oint\frac{e^{i\omega t}}{\omega(\omega_i^2-\omega^2)}.$$

This integral is easily calculated by the method of residues,

$$\theta_1(x,t) = \sum_{i=1}^{\infty}\frac{C_i\Theta_i}{\omega_i^2}\,[1-\cos\omega_i t]. \tag{6}$$

This expression gives the motion due to a sudden applied moment $M_0(x)$ as a sum of free oscillations.

In case of concentrated loads, the formula is similar. Suppose that the shaft carries n discs of moments of inertia I_k. The distributed moment $M_0(x)$ is replaced by the moments M_{0k} applied to those discs, and the distributed amplitude $\theta(x)$ by the amplitude θ_k of each disc. There will be, in general, n periods of free oscillations. The coefficients C_i are given by the following expressions

$$C_i = \frac{\sum_{k=1}^{n} M_{0k}\,\Theta_{ik}}{\sum_{k=1}^{n} I_k\,\Theta_{ik}^2}.$$

We call Θ_{ik} the amplitude of oscillation of the "k th" disc due to the

free oscillation of order "i." If all the moments are suddenly applied at the same time the angular displacement of each disc is given by

$$\theta_k(t) = \sum_{i=1}^{n} \frac{C_i \, \Theta_{ik}}{\omega_i^2} [1 - \cos \omega_i t]. \tag{7}$$

Using a well-known method, we will now deduce from this analysis the motion due to a moment of the form $M_0(x)\psi(t)$, where $\psi(t)$ is an arbitrary function of time. It may be considered as composed of an infinite number of small increments $M_0(x) \frac{\bar{d}\psi}{\bar{d}t} dt$. Each of these increments produces an oscillation given by (6) or (7) so that the total motion in case of distributed parameters is given by the integral

$$\theta(x, t) = \int_0^t \theta_1[x, t - \tau] \frac{\bar{d}}{\bar{d}\tau} \psi(\tau) \bar{d}\tau.$$

Integrating by parts and assuming $\psi(o) = 0$,

$$\theta(x, t) = \int_0^t \frac{\partial}{\partial t} [\theta_1(x, t - \tau)] \psi(\tau) d\tau,$$

$$\theta(x, t) = \sum_{i=1}^{\infty} \frac{C_i}{\omega_i} \Theta_i(x) \, [\sin \omega_i t] \int^t \psi(\tau) \cos \omega_i \tau d\tau - \cos \omega_i t \int_0^t \psi(\tau) \sin \omega_i \tau d\tau.$$

$$\text{Put } f_1(\omega_i, t) = \int^t \psi(\tau) \cos \omega_i \tau d\tau,$$

$$f_2(\omega_i, t) = \int_0^t \psi(\tau) \sin \omega_i \tau d\tau.$$

The expression (7) for $\theta(x,t)$ shows that the motion due to an applied moment of the form $M_0(x)\psi(t)$ is composed of a series of free oscillations each of which has an amplitude

$$\frac{C_i}{\omega_i} \Theta_i(x) \sqrt{f_1^2(\omega_i, t) + f_2^2(\omega_i, t)}.$$

This principle leads to the solution of the announced problem. Consider a shaft rotating with a constant acceleration. In most cases the amplitude of variation of the torque will be constant. We call v the angular acceleration and put $\beta = \frac{v}{2}$. The variable part of the moment is, per unit length,

$$M_0(x)\psi(t) = M_0(x) \sin (\beta t^2 + \varphi).$$

Put

$$F(\omega_i, t) = \sin \omega_i t \int_0^t \cos \omega_i \tau \sin (\beta\tau^2 + \varphi) d\tau - \cos \omega_i t \int_0^t \sin \omega_i \tau \sin (\beta\tau^2 + \varphi) d\tau.$$

The motion is then given by formula (7) in the form

$$\theta(x, t) = \sum_{i=1}^{\infty} \frac{C_i}{\omega_i} \Theta(x) F(\omega_i, t). \tag{8}$$

In order to calculate F we shall introduce the quantities

$$\alpha = t \sqrt{\frac{v}{\pi}} = 4 \sqrt{N_i}$$

$$\xi = t \sqrt{\frac{v}{\pi}} = \alpha \frac{N}{N_i}$$

where N_i is the number of revolutions performed when the critical speed is reached at the moment t_i, and N the number of revolutions performed after any time t.

By an elementary transformation we may write

$$F(\omega_i, t) = {}^1/_2 \sin(\omega_i t + \varphi) \int_0^t \sin(\beta\tau^2 + \omega_i \tau) d\tau + {}^1/_2 \sin(\omega_i t - \varphi) \int_0^t \sin (\beta\tau^2 - \omega_i \tau) d\tau + {}^1/_2 \cos(\omega_i t + \varphi) \int_0^t \cos(\beta\tau^2 + \omega_i \tau) d\tau + {}^1/_2 \cos(\omega_i t - \varphi) \int_0^t \cos (\beta\tau^2 - \omega_i \tau) d\tau.$$

Introducing now the quantities α and ξ,

$$\Psi(\xi, \alpha, \varphi) = 2 \sqrt{\frac{v}{\pi}} F(\omega_i, t) = \sin\left(\omega_i t - \varphi - \frac{\pi}{2} \alpha^2\right) \int_{-\alpha}^{\xi - \alpha} \sin \frac{\pi}{2} y^2 dy - \cos\left(\omega_i t - \varphi - \frac{\pi}{2} \alpha^2\right) \int_{-\alpha}^{\xi - \alpha} \cos \frac{\pi}{2} y^2 . dy + \sin\left(\omega_i t + \varphi + \frac{\pi}{2} \alpha^2\right) \int_{\alpha}^{\xi - \alpha} \sin \frac{\pi}{2} y^2 dy + \cos\left(\omega_i t + \varphi + \frac{\pi}{2} \alpha^2\right) \int_{\alpha}^{\xi + \alpha} \cos \frac{\pi}{2} y^2 dy. \tag{9}$$

This expression involves the well-known Fresnel integrals. By substitution in equation (8) we get for each normal mode an amplitude given by

$$\theta_i(x, t) = \frac{\sqrt{\pi}}{2} \cdot \frac{C_i \Theta_i(x)}{\omega_i \sqrt{v}} \cdot \Psi(\xi, \alpha, \varphi).$$

We are interested in the behavior of this amplitude as a function of time. Practically the critical speed will be reached in more than ten revolutions, so that,

$$\alpha > 4\sqrt{10}$$

take $\alpha > 12$.

Write

$$\int_0^u \cos\frac{\pi}{2}y^2dy = C(u),$$

$$\int_0^u \sin\frac{\pi}{2}y^2dy = S(u).$$

These functions have been tabulated.[1] In our assumption that $\alpha > 12$, the terms written on the second line of equation (9) are small and may be neglected. This equation then reduces to

$$\Psi(\xi, \alpha, \varphi) = \sin\left(\omega_i t - \varphi - \frac{\pi}{2}\alpha^2\right)\int_{-\alpha}^{\xi-\alpha}\sin\frac{\pi}{2}y^2dy - \cos\left(\omega_i t - \varphi - \frac{\pi}{2}\alpha^2\right)\int_{-\alpha}^{\xi-\alpha}\cos\frac{\pi}{2}y^2\bar{d}y.$$

This is the projection of a rotating vector whose rectangular components are

$$C(\xi - \alpha) + C(\alpha),$$

$$S(\xi - \alpha) + S(\alpha).$$

When ξ varies, the extremity of this vector moves on a curve plotted in the figure. The length of the curve taken as 0 for $\xi = \alpha$ has the value $\xi - \alpha$, and is a linear function of the time.

Consider the case where a great number of revolutions have to be performed before the critical speed is reached. This means that the value of α is great. The conclusions that we will draw from this assumption shall be practically true if $N_i > 12$.

The amplitude Ψ at any moment is represented by a vector having its origin in $0'$ and its end at the point of the curve corresponding to $t\sqrt{\frac{v}{\pi}} = \xi$. We see that the amplitude increases at first slowly, then very rapidly near the point of resonance 0 ($\xi = \alpha$) where the amplitude is $\frac{1}{\sqrt{2}}$. The amplitude then reaches a maximum $1.165\sqrt{2}$ for $\xi - \alpha = 1.25$ and decreases afterward in an oscillating way, down to $\sqrt{2}$.

Hence the maximum amplitude of the normal mode will be

$$\theta_i(x) = 1.165\sqrt{\frac{\pi}{2}}\frac{C_i\,\Theta_i(x)}{\omega_i\sqrt{v}}.$$

In the case of concentrated loads, the maximum vibration amplitude of each disc is

$$\theta_{ik} = 1.165\sqrt{\frac{\pi}{2}}\frac{C_i\,\Theta_{ik}}{\omega_i\sqrt{v}}.$$

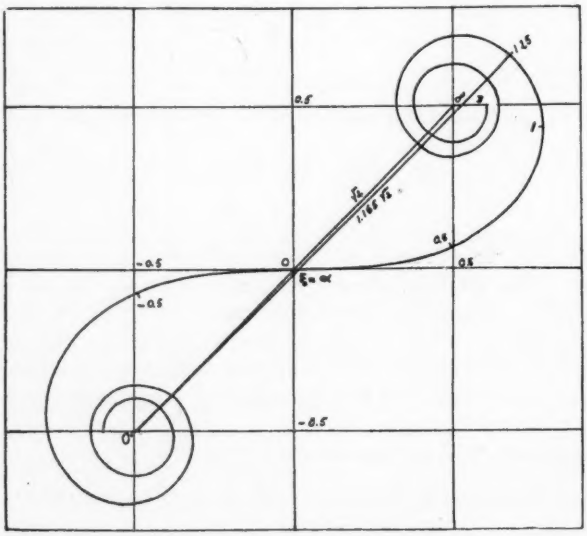
If the moment acting on the shaft is the sum of two moments applied with a phase difference φ,

$$M_{ok} = M'_{ok} \sin \beta t^2 + M''_{ok} \sin(\beta t^2 + \varphi)$$

the maximum amplitudes of the discs will be given by the same formula, provided that C_i is the vectorial sum (with a phase difference φ) of the two coefficients C'_i and C''_i corresponding respectively to the separate moments M'_{ok} and M''_{ok},

$$C_i = \sqrt{C'^2_i + C''^2_i + 2C'_i C''_i \cos \varphi}.$$

Example.—We shall apply this result to the crankshaft of a six cylinder



two cycle engine. The crankshaft is supposed to be free at both ends, and may be considered as a shaft loaded with six discs of moment of inertia I. If M is the mean torque produced by this motor, $\frac{M}{6}$ will be the mean torque applied by the pistons to each of the discs, and at the same time the amplitude of variation of this torque. We assume that the explosions occur at the same moment in pistons equidistant from the center, in the following order,

$$(1, 6) \quad (2, 5) \quad (3, 4).$$

What is the effect of acceleration on the third harmonic of this shaft? The amplitudes of free vibration of the third harmonic are

$$\begin{aligned}\Theta_{31} &= 1\\ \Theta_{32} &= -2\\ \Theta_{33} &= 1\\ \Theta_{34} &= 1\\ \Theta_{35} &= -2\\ \Theta_{36} &= 1\end{aligned}$$

Each piston causes a certain amplitude of vibration of the third harmonic, and this effect is characterized by the corresponding coefficients,

$$C_{3k} = \frac{M}{6I} \cdot \frac{\Theta_{3k}}{\sum_{K=1}^{6} \Theta_{3k}^2} = \frac{M}{F^2 I}\,\Theta_{3k}.$$

These coefficients have to be added vectorially, due to the fact that there is a phase difference in the applied moments.

Let us first add the vectors which have no phase difference:

$$C_{31} + C_{36} = \frac{M}{F2I}\,(\Theta_{31} + \Theta_{36}) = 2\,\frac{M}{F2I}.$$

$$C_{32} + C_{35} = \frac{M}{F2I}\,(\Theta_{32} + \Theta_{35}) = -4\,\frac{M}{F2I}.$$

$$C_{33} + C_{34} = \frac{M}{F2I}\,(\Theta_{33} + \Theta_{34}) = 2\,\frac{M}{F2I}.$$

These three vectors have a phase difference $\frac{2\pi}{3}$ and their vectorial sum is,

$$C_3 = \frac{M}{KI}.$$

The maximum critical amplitude of vibration of each disc is

$$\theta_{3k} = 1.165\sqrt{\frac{\pi}{2}}\,\frac{C_3\,\Theta_{3k}}{\omega_3\sqrt{\nu}}.$$

The difference between θ_{31} and θ_{32} gives the maximum torsional strain

$$\theta_{31} - \theta_{32} = 1.165\sqrt{\frac{\pi}{2}}\,\frac{M}{4I\omega_3\sqrt{\nu}}.$$

[1] E. Jahnke and F. Emde, "Funktionentafeln," p. 24.

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THE CRUSTACEAN CHROMATOPHORE ACTIVATOR AND THE GONADS OF THE RAT

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In 1928 Perkins and later Koller (1928) reported the production in the eye stalks of crustaceans of a chromatophore activator—presumably a hormone—which, carried in the blood-stream, induced chromatophore contraction resulting in pallor in these crustaceans. It was later found that the active substance was widely interspecific and also caused melanophore contraction in fishes (Koller and Meyer, 1930), and melanophore expansion in amphibian tadpoles (Perkins and Kropp, 1932). It is known also that mammalian endocrine secretions affect the melanophores of lower vertebrates. Since in some respects the action of the crustacean chromatophore activator on vertebrates is suggestive of the action of pituitary secretion, it seemed desirable to determine the action of the chromatophore activator on the ovary, testis and sexual cycle of a mammal.

The experiments consisted of the injection of varying doses of *Palaemonetes vulgaris* eye stalk extract into male and female rats twenty-one days of age. The eye stalks were extracted with 0.8 per cent NaCl solution in such a way that each 0.05 cc. of the extract contained the active substance from one eye stalk. The extract was known to contain the active material since samples produced the usual chromatophore contraction in black-adapted *Palaemonetes*. Twenty animals were grouped by fives and the members of each group were injected subcutaneously with a total of extract representing respectively five, ten, fifty and one hundred eye stalks. The injections were made over periods of three days for the two lower groups and four days for the two higher. On the fourth day following the first injection the animals were killed and compared with controls of the same age and sex for body weight, weight of gonads, and gross and histological appearance of the gonads.

No significant differences in any of these criteria were detected in either males or females. The experimental animals were wholly indistinguishable from the saline-injected controls. Despite its pituitrin-like qualities, therefore, as regards action upon melanophores, this invertebrate secretion failed to produce the anticipated alteration in the mammalian gonads.

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THE PENETRATION OF *m*-BROMO-PHENOL INDOPHENOL AND OF GUAIACOL INDOPHENOL INTO VALONIA VENTRICOSA

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This paper is a continuation of the study of the penetration of oxidation-reduction indicators into living plants of *Valonia ventricosa*. Two indicators have been used, namely, meta-bromo-phenol indophenol and guaiacol indophenol. The first one is the most positive of any of the indicators so far used in this series and is very readily reduced by living cells. The second one is in the region of neutrality of Clark's[1] oxidation-reduction scale, between Lauth's violet which penetrates *Valonia* in the oxidized form, and *o*-cresol indophenol which penetrates in the reduced form.

The method of experimentation was substantially that described in a previous publication.[2] The marine alga, *Valonia ventricosa*, was placed in solutions of the indicators and the sap subsequently examined as to its dye content. It was found that the first dye penetrates readily and is found in the sap in the reduced form. Even when the plants are very soft and the concentration of dye inside equals that in the external solution, thereby showing that the plants are injured, the dye in the sap is still in the reduced state. The second dye is more difficult to reduce. It was therefore of interest to see whether it would be found in the sap of *Valonia* in an oxidized or reduced form. This dye penetrates less readily than the other indophenol dyes, but is reduced by the plants. In a few cases, when a more concentrated solution was used, dye in the oxidized form was found in the sap, showing that it is on the border-line of oxidation-reduction in the case of *Valonia*.

The rate of penetration of both dyes depends upon the pH of the external solution, temperature being constant, but is greater for the first dye than for the second. In both cases the concentration of the dye in the sap of uninjured cells at equilibrium is less than that in the external solution.

Spectrophotometric analyses of the dye penetrating into the sap show that the dye has not been altered chemically in penetrating; that the wave length of maximum absorption in each case coincides with that for the dye in the external solution.

This is a preliminary paper. Details will be subsequently published. Grateful acknowledgment is made to the Bache Fund of the National Academy of Sciences for enabling the writer to obtain *Valonia* and to the Marine Biological Laboratory where this work was done.

[1] Clark, W. M, "Studies on Oxidation-Reduction," *U. S. Pub. Health Service, Hyg. Lab. Bull. No. 151* (1928).

[2] Brooks, M. M., *Am. J. Physiol.*, **76**, 360 (1926).

STUDIES ON THE GROWTH HORMONE OF PLANTS. II. THE ENTRY OF GROWTH SUBSTANCE INTO THE PLANT

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1. *Introduction.*—In recent years methods have been developed whereby the growth substance or hormone which controls the elongation of plant cells may be prepared. Furthermore, it has been shown that this substance may be quantitatively determined. This was first done by F. W. Went (1928) who obtained the growth substance by allowing tips of Avena coleoptiles to stand upon agar. He found that if the agar was then placed upon one side of a decapitated coleoptile, the increase in growth on that side produced a curvature which was, he found, a measure of the amount of growth substance in the agar. Since the amount of growth substance in the agar was determined by the number of tips which were placed upon it for a given time, he was able to establish that there was a proportionality between the growth substance added and the curvature produced. This proportionality held up to a certain limiting curvature which has been called the "maximum angle."

Recent studies have made it possible to obtain the growth substance in much larger amounts. Nielsen (1930) showed that it was produced by the growth of molds and Bonner (1932) studied the conditions under which it was produced by Nielsen's culture of *Rhizopus suinus*. Studies on large scale production were carried out by Thimann and Dolk (1932), using the same mold.

The partially purified extract from this source has been used in these experiments. The use of a solution of high activity, from which dilutions may be made, enables repeated tests to be carried out with accurately controlled quantities of growth substance. This procedure is much more quantitative than the method of allowing a given number of coleoptile tips to stand on agar. In a preceding communication the chemical properties of the growth substance were studied and a method of partial purification worked out (Dolk and Thimann, 1932). It is of interest that Kögl and Haagen-Smit (1931) have obtained either the same substance or one of a similar physiological action from urine and have been able to prepare a crystalline product of high activity.

In the course of experiments to elucidate the relationship between growth substance and growth in the plant, it was necessary to determine whether all of the growth substance applied to the plant actually entered, or, if not, what proportion remained in the agar. The experiments of Went (1928) seemed to show that the curvature is proportional to the

amount of growth substance in the agar block. Thus, he found that if its concentration was reduced by some factor, and the size of the block increased by the same factor, the resulting curvatures appeared to be the same. However, van der Wey (1932) found that the curvature was proportional to the *concentration* of the growth substance in the agar and nearly independent of the size of the blocks. In general, we have confirmed the finding that the curvature is proportional to the concentration of growth substance and not to the amount of it present in the block. It will be shown, however, that if it be assumed, as seems justified, that the rate of entry of the substance into the plant is proportional to its concentration in the block at each moment, then the variation of curvature with size of block is a necessary consequence, and that hence an explanation of the divergent results of different investigators is found.

2. *Methods and Materials.*—The concentrated growth substance was prepared from a culture of the mold, *Rhizopus suinus*, grown upon a large scale as described by Thimann and Dolk (1932). The filtered and evaporated medium was extracted with ether, and the product of the first three extractions, freed from ether and separated from insoluble oils, was used in these experiments.

The pure line of Avena, "Siegeshafer," which was used was kindly supplied by Dr. Åkermann of Svalöv. The quantitative assay of the extracted substance depends, as already pointed out, upon the curvature produced by the difference in growth between the two sides of a decapitated coleoptile, when one side is supplied with the growth substance. This technique was originated by Went (1928), who has also described the determination of the resulting curvature by direct measurement of the angle. The method used by Nielsen (1930) and by Boysen-Jensen (1931) to determine the curvature involves three measurements: radius of curvature, length of curved portion and thickness of the coleoptile, and is on this account more troublesome. It was also pointed out in the preceding communication that since radii of curvature greater than 8 cm. have to be neglected, the relationship between growth substance and curvature is no longer linear at high dilution. In a paper recently received, (1932), Boysen-Jensen has objected to this criticism, but it seems to us still to be valid.

The standard agar blocks were made up using the cutting instruments of Dolk (1930); each had a volume of 10.7 cubic millimeters.

3. *The Proportion of Growth Substance Entering the Plant under Defined Conditions.*—To determine what proportion of the applied growth substance actually enters the plant, agar blocks containing growth substance in suitable dilution were prepared and placed upon decapitated coleoptiles in the usual way. At the expiration of the test the curvatures were photographed, the blocks removed and placed upon another set of de-

capitated coleoptiles. After 110 minutes the curvatures were again photographed. The preliminary results, given in table 1, show that

TABLE 1

CURVATURES FROM REPEATED APPLICATIONS OF BLOCKS

FIRST APPLICATION		SECOND APPLICATION	
CURVATURE	NO. OF PLANTS	CURVATURE	NO. OF PLANTS
13.2°	12	12.2°	12
12.2°	12	8.5°	12
6.8°	6	7.2°	6

the curvatures given by blocks which have been, for a second time, 110 minutes upon coleoptiles, are not greatly smaller than those given the first time. Since these results are the reverse of those which would be expected according to Went, the experiments were repeated using growth substance obtained from the tips of Avena coleoptiles. This was done to preclude the possibility that the behavior of this preparation might be different from that of the growth substance obtained from Rhizopus. Tips were therefore cut off, placed upon agar, the agar left 30 minutes to allow for the attainment of uniform distribution, divided into twelve and tested as before. Table 2 shows that the results, although somewhat irregular, were essentially the same, i.e., while a curvature was produced at the second application, it was distinctly less than that produced at the first.

TABLE 2

EXPERIMENTS WITH THE GROWTH SUBSTANCE FROM AVENA TIPS

	TIME ON AGAR	FIRST APPLICATION		SECOND APPLICATION	
NO. OF TIPS	MINUTES	CURVATURE	NO. OF PLANTS	CURVATURE	NO. OF PLANTS
8	85	3.4°	10	3.5°	10
16	90	6.3°	10	6.0°	9
14	120	9.8°	12	7.5°	11
30	122	11.7°	12	8.5°	10

The next step was to measure as accurately as possible the decrease in curvature at the second application. For this purpose a standard growth substance solution was made up and assayed by the curvature method. The curvatures produced at the first and second applications were then measured upon a large number of plants. Table 3 shows that the fractions of the growth substance which leave the block at different concentrations are reasonably constant, (13–21%), particularly when it is considered that the quantity involved is a rather small difference between two determinations which are both subject to some variation.

In the case of the experiment at 19.3 units concentration, the initial curvature was calculated from assay at lower concentration, and the second curvatures were very close to maximum angles and are therefore somewhat low. This experiment is therefore not included in the mean.

4. *Theoretical Considerations.*—Let it be first assumed that:

The rate of gain of growth substance by the plant is proportional at any moment to its concentration in the agar block at that moment..... (1)

TABLE 3

DECREASE OF CURVATURE FROM REPEATED APPLICATION OF BLOCKS

CURVATURE FIRST APPLICATION	CURVATURE SECOND APPLICATION	PERCENTAGE DECREASE	NO. OF PLANTS
9.0°	7.6°	16%	20
9.0°	7.7°	14%	20
9.5°	8.3°	13%	22
9.2°	7.7°	16%	19
15.2°	12.5°	18%	20
14.1°	12.0°	15%	14
13.9°	12.1°	13%	14
* 19.3°	15.7°	19%	23
* 19.3°	15.2°	21%	25
		Mean of first seven determinations = 15%	

* Assay at lower concentration.

This assumption is not in conflict with the views of any investigator in this field, and it will be shown that the results to which it leads, both in regard to repeated applications of the same block, and in regard to the variation in curvature with size of block, are in accord with those obtained by experiment.

Let the concentration of the substance in the block at any moment be x/v, where x is the amount in grams, and v the volume of the block. Then from (1),

$$dx/dt = - k\,x/v. \qquad (2)$$

On integration this becomes

$$\ln x_1/x_2 = k(t_2 - t_1)/v.$$

In experiments carried out, as were ours, over a constant period of time, (110 minutes) we can put $k(t_2 - t_1)/2.303$ as a new constant K, i.e.,

$$\log x_1/x_2 = K/v. \qquad (3)$$

Now the excess growth produced by the growth substance must be proportional to the amount of substance entering the plant, or, in the case of unilateral growth, the curvature θ is proportional to Δx, the amount of growth substance which enters the plant in the 110 minutes allowed, i.e.,

$$\theta = c(x_1 - x_2) \qquad (4)$$

where x_1 is the amount initially in the block, x_2 the amount remaining in the block at the end of the test and c the proportionality constant.

Now consider the case of repeated application of a block of volume v. Equations of the type (3) and (4) may be written for each application:

$$\log (x_1/x_2) = K/v = \log (x_2/x_3)$$

and

$$c(x_1 - x_2) = \theta_1; \quad c(x_2 - x_3) = \theta_2. \tag{4a}$$

Hence

$$x_2 = \frac{x_1}{\text{antilog } K/v}; \quad x_3 = \frac{x_1}{(\text{antilog } K/v)^2}.$$

Substituting for x_2 in equation (4a)

$$\theta_1 = cx_1\left(1 - \frac{1}{\text{antilog } K/v}\right). \tag{5}$$

Similarly, substituting for x_3,

$$\theta_2 = \frac{cx_1}{\text{antilog } K/v}\left(1 - \frac{1}{\text{antilog } K/v}\right). \tag{6}$$

Hence

$$\theta_1/\theta_2 = \text{antilog } (K/v). \tag{7}$$

From the mean value given in table 3, the percentage decrease in curvature is 15%, i.e., the ratio between the angles at the first and second applications is 100/85, or

$$\theta_1/\theta_2 = 100/85 = \text{antilog } (K/v).$$

From this

$$K/v = \log 1.177 = 0.0709. \tag{8}$$

The value of K/v is independent of the concentration of growth substance and size of block, and is a characteristic constant for the reaction here considered.*

The quantity x_1, the amount of growth substance originally present in a block giving a curvature θ, is of some interest. For the block giving a curvature at the first application of 9°, its value is $60°/c$, where c is the relation between grams and degrees. This means that with an infinite number of tests the sum of the curvatures obtained would be 60°.

5. *Influence of Size of Block.*—If now the volume of the block be reduced from the size first assumed, v_0, to $1/n$ of this, then the right-hand side of equation (2) becomes multiplied by n, and hence the rate at which the amount in the block changes is correspondingly increased. Thus, the smaller the block, the larger the percentage of its growth substance which will pass out during the test. The ratio θ_1/θ_2 will therefore be increased for smaller blocks, i.e.,

$$\theta_1/\theta_2 = \text{antilog } (Kn/v_0). \tag{9}$$

It might be objected that a decrease in the volume of the block is accompanied by a decrease in the area of contact between block and coleoptile; however, the accompanying sketch, to scale, figure 1, indicates that even with blocks one-eighth of our standard size, the area of contact is completely unchanged. It is not necessary, therefore, to consider changes in the area of contact in these experiments.

However, it has been found that if the block be placed flat on a coleoptile from which the primary leaf has been pulled out, the rate of entry of the growth substance is more than doubled. In this case the contact area is correspondingly more than twice that shown in the left-hand diagram.

From considerations similar to those above, it is possible to calculate the curvature which would be expected from a block of any given size, provided that the curvature resulting from the use of a block of another size, containing the same concentration of the growth substance, is known.

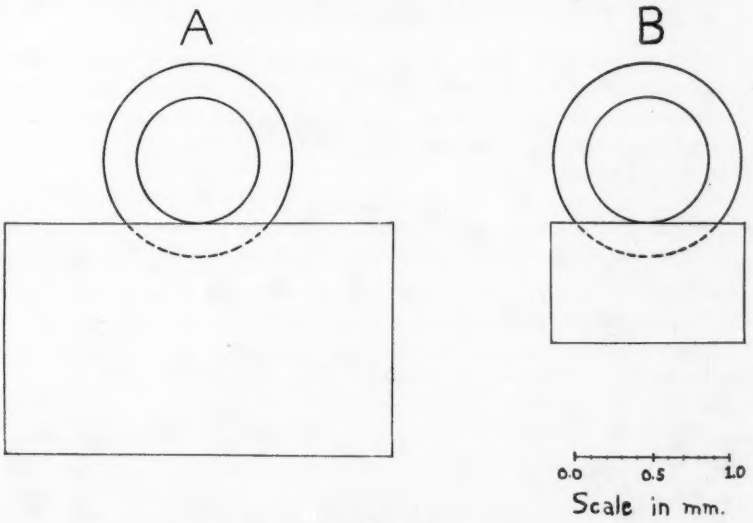

FIGURE 1

Agar blocks resting unilaterally upon decapitated coleoptiles, and in contact with the primary leaf. A. Standard size block. B. One-eighth block. The area of contact is the same in each case.

When a block of our standard size, v_0, is used, the resulting curvature may be designated θ_0. From equation (5)

$$\theta_0 = cx_0\left(1 - \frac{1}{\text{antilog } K/v_0}\right)$$

or, putting in the value of K/v_0 obtained above,

$$\theta_0 = 0.150cx_0 \qquad (10)$$

It might have been foreseen that, since θ is proportional to x, and since successive θ's decrease by 15%, therefore successive x's decrease by 15%, and hence the amount passing into the plant, which is responsible for the curvature, is proportional to 15% of x.

The amount of growth substance in blocks having the concentration above, but volume v_0/n, is $x_1' = x_0/n$, and the angle obtained at the first application is given by the equation

$$\theta_1' = \frac{\theta_0}{0.150n}\left(1 - \frac{1}{\text{antilog } Kn/v_0}\right). \qquad (11)$$

Similarly the angle for a second application can be obtained:

$$\theta_2' = \frac{\theta_0}{0.150n \cdot \text{antilog } Kn/v_0}\left(1 - \frac{1}{\text{antilog } Kn/v_0}\right). \qquad (12)$$

With the aid of equations (11) and (12), the curvatures θ_1' and θ_2' caused by blocks having a volume any fraction $1/n$ of our standard blocks, may be compared with the curvature θ_0 caused by blocks of the standard volume and of the same initial concentration. Table 4 shows the values of θ_1' and θ_2' as a factor of θ_0.

TABLE 4

COMPUTED RATIOS FOR THE CURVATURES OBTAINED WITH FRACTIONAL BLOCKS TO THOSE OBTAINED WITH WHOLE BLOCKS. VALUE OF K/v TAKEN FROM (8)

RATIO OF BLOCK SIZE, n	ACTUAL VOL. OF BLOCK, MM.^{3}	ANTILOG $\frac{Kn}{v_0}$	$\frac{\theta_1'}{\theta_0}$	$\frac{\theta_2'}{\theta_0}$
1	10.7	1.18	1	0.85
2	5.35	1.39	0.915	0.66
4	2.68	1.93	0.790	0.41
8	1.34	3.73	0.598	0.16
9	1.19	4.43	0.573	0.129
10	1.07	5.23	0.539	0.103
12	0.89	7.29	0.479	0.066

For experimental determinations of these curvatures, the 12 blocks made by the special cutter, each of the volume 10.7 mm.^{3}, were further subdivided into 2, 4 and 8. This was done by hand, using the cutter blades without the socket, and to allow for inequalities in the resulting blocks, as many as possible of them were used and the curvatures averaged. Thus, if one of the originally equal blocks were divided into two unequal halves, both halves would be used and the inequality balanced out.

With these small blocks the risk of drying out during the test is much greater. This may lead to anomalous results, as pointed out by Went (1928). Special precautions were therefore taken, and a high humidity was successfully maintained without guttation.

The collected results are given in table 5, together with the expected curvatures calculated from the ratios in table 4. Considering that the curvatures are only measured to a degree, and that the fractions are obtained by averaging a number of plants (in general from 12 to 24), we consider the agreement between observed and calculated values to be fairly satisfactory.

It is thus interesting to observe that a relationship derived from experiments with repeated application of the same block can be used to predict the curvatures obtained from a single application of fractional blocks.

TABLE 5

CURVATURES OBTAINED WITH FIRST AND SECOND APPLICATIONS OF FRACTIONAL BLOCKS

EXP. NO.	CURVATURE WITH STANDARD BLOCK, θ_0	SIZE OF FRACTIONAL BLOCK	CURVATURE WITH FRACTIONAL BLOCK: FIRST APPLICATION OBSERVED θ'_1	FIRST APPLICATION CALCULATED θ'_1	SECOND APPLICATION OBSERVED θ'_2	SECOND APPLICATION CALCULATED θ'_2
4	9.6°	Half	9.2°	8.8°	..	..
5	13.8°	Half	12.6°	12.6°	..	..
	13.8°	Quarter	10.6°	10.9°	..	..
	13.8°	Eighth	9.1°	8.2°	..	..
6	14.4°	Half	12.9°	13.0°	9.0°	9.5°
	14.4°	Quarter	9.0°	11.4°	6.5°	5.9°
	14.4°	Eighth	7.9°	8.6°	1.5°	2.3°
7	13.5°	Eighth	9.0°	8.1°	3.2°	2.2°
8	14.4°	Half	13.2°	13.0°	9.6°	9.5°
9	15.6°	Eighth	8.5°	9.3°	2.5°	2.5°

6. *Comparison with the Results of Other Workers.*—The above considerations enable our results to be compared with those of Went. In order to find, as he did, that the curvature is proportional to the *amount* of growth substance in the block, the conditions must be such that all of the substance enters the plant. Since Went's *Avena* plants were from the same stock as ours, and since he allowed the same time (110 minutes) for the curvatures to develop, the values of K and t are the same, and hence the difference between his results and ours must be sought in a difference in the volume of the blocks. In general, the blocks of Went were much smaller than ours. His largest and smallest blocks had only 0.127 and 0.065, respectively, of the volume of our standard block (10.7 mm.^{3}).

Went presents a number of experiments designed to show that the curvature obtained was proportional to the *amount* of growth substance in the agar. In his determinations, which are summarized in table 6,

TABLE 6

COMPARISON BETWEEN WENT'S OBSERVATIONS AND OUR CALCULATIONS

DATA OF WENT				COMPARISON		
EXP. NO.	CONCN. OF GROWTH SUBSTANCE INITIALLY IN BLOCK	VOLUME OF BLOCK, MM.^{3}	ANGLE	VOL. OF WENT'S BLOCK / VOL. OF OUR BLOCK	RATIO BETWEEN ANGLES OBSERVED	RATIO BETWEEN ANGLES CALCULATED
176	a	0.90	9.1 ± 0.4	0.083	1.52 ± 0.12	1.67
180	$a/2$	1.20	6.0 ± 0.5	0.111		
181	b	1.20	8.3 ± 0.6	0.111	0.91 ± 0.07	0.80
182	$3b/2$	0.90	9.1 ± 0.6	0.083		
190	c	0.90	8.6 ± 0.5	0.083	1.34 ± 0.14	1.21
194	c	0.69	6.4 ± 0.7	0.065		
333	d	0.90	7.2 ± 0.6	0.083	1.43 ± 0.20	1.28
334			7.1 ± 1.0			
335	d	1.35	10.3 ± 0.4	0.127		

the different concentrations of growth substance were obtained by varying the number and diffusion time of coleoptile tips upon agar. For clarity we have expressed these in the table simply by the ratios of the concentrations used in each set of experiments. The first four columns give his essential data. Column 5 gives the ratio of volume between his blocks and our standard blocks, and columns 6 and 7 present the ratios between the angles, as observed by him, and as calculated by equation (11) using the factors in table 4.

It will be seen that the agreement, though not exact, is nevertheless fairly close. There is no systematic difference between the calculated and observed ratios, and since the determinations of Went were made with only 9 or 12 plants in each case, they must, however carefully carried out, have been subject to considerable variation. It is therefore clear that the results of Went, which previously had seemed to show that the curvature was proportional to the *amount* of growth substance, are equally in accord with our interpretation. The results of van der Wey, recently reported, seem to show a still smaller dependence of curvature upon block size than would be expected from our calculations. His experiments, however, were carried out using the method of double decapitation previously described by him, and under these circumstances it may well be that the numerical relations deduced above do not apply. His results, however, confirm ours qualitatively in that they show that only a small fraction of the growth substance in a block diffuses out at a single application.

Summary.—(1) An attempt has been made to provide a theoretical foundation for the process by which the growth-promoting hormone or growth substance passes from solution in agar into decapitated coleoptiles of *Avena*.

(2) By applying an agar block containing growth substance to a succession of test plants and measuring the resulting curvatures it has been found that a constant fraction of the growth substance in the block passes into the plant.

(3) Upon the assumption that the amount of growth substance entering the plant is proportional to its concentration in the agar, our data make possible a calculation of the reaction rate constant for the process. This constant is independent both of the concentration of growth substance and of the size of block.

(4) For blocks of different sizes, curvatures calculated from the value of this constant agree closely with those observed.

(5) The data of Went, which were originally believed to indicate a different behavior of the growth substance, are shown to agree, within the limits of experimental error, with the mechanism here described.

The authors take this opportunity of thanking Mrs. F. Dolk for her untiring and skilful assistance.

* It has, however, been shown by Du Buy that the curvatures obtained are reduced when higher concentrations of agar are employed. Our value of K/v applies to 1.5% agar.

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IMPULSES FROM SENSORY NERVES OF CATFISH

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Using a Matthews[1] amplifier and oscillograph in conjunction with a loud speaker, a standing wave screen and a camera, electrical responses were recorded from various sensory nerves of the catfish *Ameiurus nebulosus* (Les.) in response to mechanical, thermal and chemical stimulation of the receptors which these nerves supply. The nerves tested were lateral line nerves, spinal nerves supplying the skin of the flank, and branches of the facial nerve supplying the tactile endings and tastebuds of the lips and barbels. Some 52 nerves were examined in 32 fishes.

After severing the medulla oblongata (in some cases brain and cord were pithed), the nerve to be tested was exposed at a point as near its region of emergence from the central nervous system as was anatomically convenient, and freed from surrounding tissues for a suitable length. It was then tied proximally with a thread, cut and drawn across silver, silver chloride electrodes connected to the recording system. The preparation was generally so arranged that the receptive areas were kept immersed in water while the nerve and incision were out of water and bathed with Ringer solution.

Maps of skin areas supplied by spinal and facial nerves were made by stroking the skin with the tip of a feather and listening to the bursts of impulses from the loud speaker. In this way it was possible to ascertain the particular skin areas supplied by a branch of nerve. By means of the volume of sound, it was also possible to determine the relative distribution of tactile sensitivity of the skin.

The impulses set up in facial and spinal nerves in response to touching the skin were typically the A-type impulses of Erlanger and Gasser[2]—rapid impulses of from 10 to 20 microvolts. The lips and barbels were found to be very much more sensitive to mechanical stimulation than was the skin of the flank. Bursts of impulses were initiated from the lips and barbels in response, not only to direct contact, but also to small ripples in the water bathing the skin. Impulses were initiated by a stream of water a few centimeters per second projected toward the skin beneath the surface of the bath.

Maps of the cutaneous areas supplied by the facial nerves on the two sides of the head showed complete bilateral symmetry for the distribution of sensitivity. This symmetry of nerve supply along with the great sensitivity of the lips and barbels to streams of water furnishes a basis for rheotropic orientation according to Loeb's well-known mechanism of bilateral equilibration of central excitatory states (cf. also Jordan[3]).

The barbels of *Ameiurus* as well as the lips, mouth and skin of the flank possess numerous typical tastebuds[4,5,6,7] and experiments involving responses to dissolved materials indicate clearly that these organs are extremely sensitive to sapid substances in the water.[5,8]

Solutions of substances were allowed to diffuse slowly into the water near the skin. Solutions of acetic acid ranging in concentrations from 1 *per cent* to 20 *per cent*, a 10 *per cent* solution of NaCl and a concentrated sugar solution were used as stimuli. In addition, juice pressed from meat which normal catfish devour voraciously was tested. These solutions were allowed to diffuse very slowly against immersed lips and barbels which, before, during and after chemical stimulation were highly responsive to mechanical stimuli. With maximal amplification very indistinct impulses were visible on the screen and were audible from the loud speaker in response to all of the reagents, excepting the sugar solution. These impulses were not produced by tap water or by Ringer solution. Adaptation to the stimulus took place after a few minutes' exposure, but was renewed by reapplication of the solution, except in cases where strong concentrations of acetic acid (75 *per cent*) had been used initially. This treatment evidently anaesthetized the tastebuds after a minute's exposure.

The marked specificity in magnitudes of potential of the impulses for touch and taste from the same nerve (the "taste" impulses were always less than 10 *per cent* of the amplitude of those set up by touch) may possibly be correlated with the size of the fibres supplying touch and taste receptors.[9] In *Ameiurus*, according to Herrick,[6] the tastebuds are primarily innervated by communis components of the 7th cranial nerve. The communis fibres take origin from the geniculate ganglion. The tactile receptors are of the free ending type and are supplied by fibres from the

Gasserian ganglion. The cells of the geniculate ganglion are all small, while those of the Gasserian ganglion are, for the most part, large.

Changes of temperature of the water bathing the skin to various levels, between 0° and 28°C., produced no detectable nerve impulses in spinal or facial nerves.

The lateral line system of *Ameiurus* has been described by Herrick[6] and Brockelbank.[10] (Cf. also Parker[11, 12].) The neuromasts or sensory cells lie at intervals along the lateral line canal between the pores which mark exteriorly the course of the canals along the flank. The sensory cells have, at their distal ends, tufts of hair-like processes which project into the lumen of the canal.

Tests of the lateral line nerve showed that the nerve appears to be in a state of continuous spontaneous activity. This was true for 29 out of 32 nerves examined. Since the three inactive nerves failed to respond to any form of stimulation whatever, I have concluded that they were altogether nonfunctional, perhaps as a consequence of the operation. The spontaneous discharge continued unabated, in some preparations, for several hours. The impulses were diphasic but could be rendered monophasic by crushing the nerve under the distal electrode. Anaesthetization of either the nerve or the neuromasts resulted in cessation of the discharge. Lowering the temperature of the nerve in addition, slowed the rate of conduction of the impulses.

The discharge was increased during the direct application of pressure to the skin over the lateral line canal. Ripples in the water caused a corresponding fluctuation of the frequency of discharge. In some preparations slow spinal reflex swimming movements persisted when the trunk was immersed in water. These movements were accompanied by increased frequency of discharge from the lateral line nerves, probably due to pressure on the neuromasts from surrounding tissues.

Parker and Van Heusen[13] found by means of nerve cutting experiments that the lateral line organs of *Ameiurus* act as mechano-receptors and among other things they respond to low vibration-frequencies of a submerged telephone, up to approximately 350 dvs. per second. To test this effect I pressed activated tuning forks of 100, 200 and 250 dvs. against the outside of the dish containing the preparation. In about 40 *per cent* of the cases no effects were obtained but in the majority of experiments photographs of the responses showed that the randomly discharging neuromasts were apparently synchronized and thrown into a rhythmic beating by the fork. This rhythm varied between 40 to 70 beats per second and never corresponded to the rhythm of the fork, being relatively constant for a given preparation. The synchronization passed with the cessation of the vibrations of the fork, the nerve fibers again discharging at random. Other nerves failed to give this synchronous

effect with tuning forks. Cutaneous nerves were made to discharge massively by pressure on the skin. The type of discharge could thus be made temporarily similar to that of the spontaneously firing lateral line nerve. Tuning forks applied to the dish during these discharges never produced synchronization of the response, indicating that the effect was characteristic of the lateral line system and not a rhythmicity set up in the recording system.

The frequency of the spontaneous discharge was found to depend upon temperature between the experimental ranges of 0° to 28°C. By lowering the temperature of the bath in which the preparation was immersed the frequency of discharge was markedly reduced; raising the temperature produced a recovery in the frequency. A quantitative analysis of these records is now in preparation. Since the lateral line system is directly responsive to small temperature changes, and since facial and spinal nerves do not emit impulses in response to mild thermal stimulation, I am led to believe that the neuromasts might act as thermal receptors. According to experiments of Wells[14] normal catfish are very responsive to temperature differences.

The spontaneous activity of the neuromasts is unlike the behavior of any other sensory cells tested, so far as I know. It is possible that the hair-like filaments projecting into the lumen of the canal may actually be cilia which by beating set up a state of continuous excitability in the end-organs of which they are a part. There seems to be no direct evidence concerning this point one way or the other.

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ACCOMMODATION COEFFICIENT OF GASEOUS IONS AT CATHODES

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The behavior of a cathode in an ionized gas can be studied from three principal points of view: first, electrical relations described by the application of Poisson's equation to the surrounding region; second, thermal relations described by the application of the energy principle to the various processes which develop or absorb heat at the cathode surface; third, pressure reactions which are described by the application of the principle of conservation of momentum at the cathode surface.

The electrical relations were first pointed out by J. J. Thomson[1] and later made more specific and greatly extended by Langmuir and his collaborators.[2] From these considerations, applied to current-carrying and to exploring electrodes, we have gained a nearly complete picture and interpretation of the phenomena occurring between the electrodes. Of equal importance with this, however, is an understanding of the phenomena occurring at the electrode surfaces, especially at the cathode. For this study we need more information than can be gained from Poisson's equation. We need to know, for example, what fraction of the current at the cathode is carried by electrons emitted from it; what is the mechanism responsible for this emission; if this emission is of thermionic origin, what factors maintain the requisite high temperature, etc. Since there are several unknown quantities, we obviously approach the solution by investigating from several independent points of view, so as to get several independent equations. It is particularly for this reason that studies of energy and pressure relations at a cathode have considerable significance. The most complete analysis of these relations, thus far made, is in a recent paper by the author[3] which, while directed particularly at the problem of the mercury arc, is nevertheless generally applicable in principle.

Attempts to investigate cathode phenomena by studying its heat balance

were begun about ten years ago[4] but conclusions based on them have largely been vitiated through their failure to recognize the existence of an "accommodation coefficient" for ions neutralized at a cathode surface. This phenomenon was discovered by Compton and Van Voorhis[5] as a result of their observation that the heating of a cathode, bombarded by positive ions of known kinetic energy, was less than expected, even after making due allowance for complicating effects such as secondary electron emission. The interpretation is as follows:

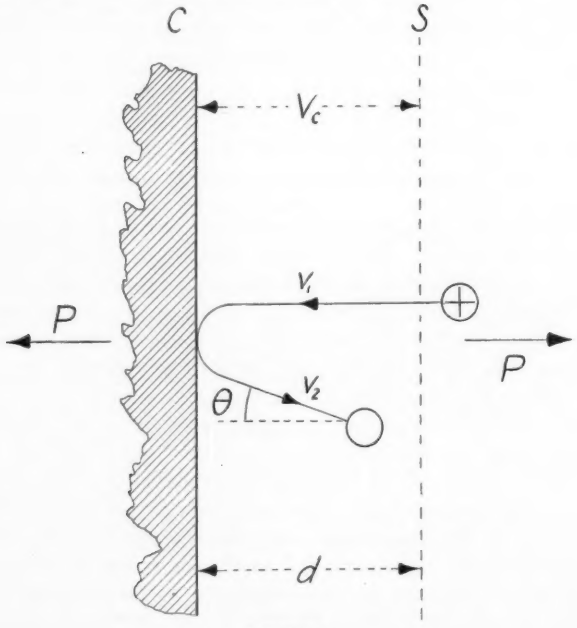

FIGURE 1

In figure 1 let C represent the cathode surface and S the boundary of the positive ion space charge sheath which always surrounds a cathode. The drop in potential V_c between the surrounding gas and the electrode is concentrated within this sheath, whose thickness we will call d. A positive ion, drifting across S, is pulled to the cathode by the field and strikes with kinetic energy $E_1 = eV_c$ and momentum $Mv_1 = (2eMV_c)^{1/2}$. The point which had been previously neglected was that this ion, after neutralization, may leave the cathode with an appreciable energy $E_2 = {}^1/_2 Mv_2^2$. Thus there is a heating effect on the cathode of $H = E_1 - E_2$ and an impulse delivered to it of $Mv_1 + Mv_2 \cos\theta$. Of this impulse, however, the term Mv_1 is exactly compensated by the mutual pull between

cathode and ion during the ion's approach, so that only the term $Mv_2 \cos z$ contributes to pressure on the cathode.

These effects may be described in terms of the "accommodation coefficient" α which has been used to describe the energy transfer of gas molecules at temperature T_1 striking a surface of temperature T_s and leaving with temperature T_2, according to the definition $\alpha = (T_1 - T_2)/(T_1 - T_s)$. If the molecules, during contact, come into thermal equilibrium with the surface, $\alpha = 1$; if the molecules rebound elastically, $\alpha = 0$. Table 1 gives some values of α, thus obtained from thermal measurements.

TABLE 1

ACCOMMODATION COEFFICIENTS OF MOLECULES

GAS	METAL	METAL TEMPERATURE	α	SEE FOOTNOTES
H_2	Pt	20°C.	0.26	[6]
		−100	0.25	[7]
		+200	0.15	[7]
	W	+1500	0.19	[8]
He	Pt	−100	0.49	[7]
		+200	0.37	[7]
		~20	0.34	[9]
	W	~20	0.06	[10] clean
		~50	0.53	[11] dirty
		~50	0.17	[11] clean
N_2	Pt	~20	0.87	[6]
	W	+1500	0.60	[8]
Ne	Pt	~20	0.65	[9]
A	Pt	~20	0.86	[9]
	W	~50	1.00	[11] dirty
			0.82	[11] clean

As might be expected, the values are extremely sensitive to the condition of the surface, such as films or roughness causing multiple impacts. However, so far as the data go, they indicate increasing accommodation coefficients with increasing molecular weights.

In the case of ions, the accommodation coefficient is $\alpha = (E_1 - E_2)/E_1$. From thermal measurements Compton and Van Voorhis[5] found the accommodation coefficients given in table 2. Considering the uncertainties

TABLE 2

ACCOMMODATION COEFFICIENTS OF IONS ON MO, DETERMINED THERMALLY

He	0.35	for $35 < V_e < 51$ volts
	0.55	for $111 < V_e < 141$ volts
Ne	0.65	for $21 < V_e < 141$ volts
A	0.75	for $21 < V_e < 141$ volts

of surfaces, these values are suggestively like those of the corresponding neutral molecules. In fact we should expect the ions to behave as neutral

molecules (except as influenced by their higher velocities) since they are presumably neutralized during contact with the cathode.

Three years ago Tanberg[12] reported measurements of pressure against the cathode of a Cu arc in a partial vacuum, and also pressure against a neighboring vane by a stream of neutral particles moving away from the cathode. Since the copper cathode was slowly volatilized, Tanberg assumed that these pressures were due to the impulsive reactions of the evaporating Cu atoms. This assumption led to the conclusion that this copper evaporated at a temperature of about 500,000°C.! The author, however, pointed out that Tanberg's pressures could be explained by momentum transfers by ions neutralized at the cathode surfaces, if reasonable values were assumed for their accommodation coefficients.[13] The decisive test of such an explanation, as contrasted with Tanberg's, would obviously be given if similar pressures were observed with non-volatilizing cathodes. This has actually been done[14] resulting in confirmation of the author's theory of cathode pressures as well as in a new method of measuring accommodation coefficients and, incidentally, the fraction of cathode current carried by electrons. As these results will be published elsewhere in detail, only those parts which are significant for the present subject will be summarized as follows:

The cathode was a molybdenum vane, about 1 cm. square, with its back protected by an insulating cover of glass, serving as the bob of a pendulum. This pendulum bob vane was suspended in a large helium arc tube, carrying several amperes at a gas pressure of the order of 1 mm. The deflection of the pendulum, when positive ions were drawn to it as a cathode, permitted the pressure against it to be computed. Neglecting for the present explanatory purposes two corrections for complicating phenomena (one of which gave us the fraction of current carried by electrons), the pressure is readily shown to be

$$P = I_+(MV_c/2e)^{1/2}(1 - \alpha)^{1/2} \qquad (1)$$

if the neutralized ions rebound with equal probability in all directions like light from a matt surface. If the rebound is always normal to the surface, the above expression should be multiplied by 2. The former assumption seems more probable, and gives a slightly more consistent description of the experiments, but this aspect of the problem still needs further examination.

The accommodation coefficient thus calculated from the pressure on a molybdenum cathode in ionized helium is shown in table 3. Two sets of values are given, depending on whether the heat of neutralization of a positive ion of zero velocity at the surface is equal to zero or equal to the difference between the ionizing potential of the gas, V_i, minus the electron

TABLE 3

ACCOMMODATION COEFFICIENTS OF He IONS ON Mo DETERMINED BY PRESSURE

V	$\alpha(\varphi_+ = 0)$	$\alpha(\varphi_+ = V_i - \varphi_-)$
35	0.681	0.463
65	0.570	0.480
95	0.512	0.449
125	0.410	0.365

work function φ_-. The uncertainty here depends on how much of the available energy in neutralization may be radiated away or carried away by the neutral atom in an excited state. The latter phenomenon is known to occur with a high degree of probability in the case of helium ions neutralized by glancing contact with a metal surface. Unfortunately all experimental attempts to measure φ_+ have thus far not been sufficiently devoid of other complications to enable a definite value to be set between these extreme limits.

In comparing the values in tables 1, 2 and 3 it is seen that they agree in order of magnitude and that the agreement between the three methods is about as good as that between different observers using one method, cf. table 1. Precise agreement is scarcely to be expected since the phenomenon is one, like electron emission or optical reflection, which depends on the condition of the metal surface and would therefore be expected to vary with films of gas or other impurity, and with the degree of roughness of the surface. Direct evidence of this is found in the experiments of Michels,[11] quoted in table 1.

Interpretation of Accommodation Coefficient.—As pointed out above, we should expect the phenomena which determine accommodation coefficients to be identical for neutral molecules and for neutralized ions, since the two particles are identical as they leave the solid surface. With ions, however, we have much higher velocities (effective temperatures) than with gas molecules, and we have them striking the surface normally instead of isotropically as regards direction. Both of these peculiarities would be expected somewhat to increase the accommodation coefficient of ions over that of molecules.

The following interpretation of accommodation coefficients appears satisfactorily to account, in a qualitative way, for such observations as have thus far been made.

Zener[15] has recently developed a quantum theory of the accommodation coefficient according to which the loss of energy of a molecule rebounding from a solid surface is computed by summing up the probabilities of transition of energy to the quantum states associated with the various degrees of freedom of the solid. For simplicity he carried through the calculations only for the degree of freedom normal to the surface. In dealing with ions, however, whose effective temperature greatly exceeds

that of the solid, it would appear legitimate to apply the simpler classical principles, for exactly the same reasons that these principles may be applied at high temperatures to specific heats.

The simplest classical picture of the phenomenon is that of an elastic impact between the impinging molecule, of mass M and a surface atom of mass M_s. Rebound occurs only if

$$M_s > M \tag{2}$$

and the average energy of such rebounding molecules is

$$E_2 = \frac{M^2 + M_s^2}{(M + M_s)^2}(E_1 - E_s) + E_s \tag{3}$$

whence the accommodation coefficient is*

$$\alpha = \frac{2MM_s}{(M + M_s)^2} \text{ subject to } M_s > M. \tag{4}$$

If we extend these ideas to include the possibility of rebound of molecules after n collisions with surface atoms, we find that the accommodation coefficient α_n appropriate to n collisions is given by

$$\alpha_n = 1 - (1 - \alpha)^n, \tag{5}$$

so that the effect of multiple collisions (as with a roughened surface) is to make the accommodation coefficient approach unity. In general, molecules impinge some once, some twice, some thrice, etc., so that the actual accommodation is a weighted mean of values calculated from Eqs. (4) and (5) with $n = 1, 2, 3 \ldots$etc. We have no means of knowing the distribution of n, so that we can only say that Eq. (4) gives a lower limit to α and that α should increase with roughness of surface.

The most striking results of this theory are (1) the conclusion that there is no rebound if $M_s < M$. We should thus expect the accommodation coefficient to equal unity (or *nearly* so, owing to the crudeness of the theory) if the mass of the ion exceeds that of the surface atom and (2) the conclusion that the accommodation coefficient is smallest for light ions.

Conclusion (1) is supported by the fact that there is no evidence of a pressure against the cathode of a carbon arc[17] such as would be caused by the existence of an accommodation coefficient less than unity for the positive ions (which in this case would be of masses equal to or greater than those of surface atoms) and by the fact that the covering of a tungsten wire with naturally occurring impurities, presumably oxide and oxygen layers, raised its accommodation coefficient for argon atoms from 0.82 for clean tungsten to unity.[11] Furthermore, pressure against the cathode of copper and iron arcs in air[17] are also so small that there is no evidence of effect of accommodation coefficient, and this again may be due to the

fact that here the cathode surface is covered by layers of oxide so that the outer oxygen layer is composed of atoms with mass equal to or less than those of the bombarding ions.

Mr. Lamar has pointed out to the writer that the absence of pressure in these arcs at atmospheric pressure may also be due to the compensating effect of holding back the atmospheric pressure from the cathode spot—a compensation which could not occur if the cathode were of large surface area, but which can occur as the result of convection currents set up around the cathode if this is of relatively small dimensions. On this assumption Lamar was able to calculate the area of the cathode spot of a copper arc to the right order of magnitude and he is now undertaking an experimental test of this suggestion.

Conclusion (2) is supported by all evidence at present available.

These considerations suggest interesting further experiments on the relation of M/M_s to α, with particular practical interest in the behavior of mercury ions, which is now being investigated.

* This assumes a *single* collision (as in Zener's theory) but allows for random distribution of direction. Had "head on" collisions only been considered, the factor 2 would have been replaced by 4. These considerations were first advanced by Baule.[16]

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RECURRENT GEODESICS ON ANY CLOSED ORIENTABLE SURFACE OF GENUS ONE

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The existence of non-periodic recurrent geodesics on certain surfaces of negative curvature has been proved by Morse.[1] Birkhoff[2] has indicated how the methods of Morse may be used to prove the existence of non-periodic recurrent geodesics on closed orientable surfaces of genus greater than one and of negative curvature. The method used is to characterize the geodesics by unending symbols and then by the construction of a recurrent symbol, a recurrent geodesic is obtained. The property of negative curvature plays an essential part in the derivation of these results.

In the case of genus one the surface cannot, of course, have everywhere negative curvature, and any characterization of geodesics by unending symbols does not seem to be profitable. Nevertheless, it can be proved in this case, and this is the purpose of this note, that there exists a non-denumerable class of non-periodic recurrent geodesics. The proof is based on the work of Birkhoff[2] and an earlier paper by the present writer.[3]

In the earlier paper it was shown that under suitable conditions on the class of the functions defining the surface, the geodesics could be studied as the geodesics on a two-dimensional Riemannian manifold with periodic coefficients and the problem could be conveniently considered in a plane divided into congruent rectangles. One of the results then derived was that given any straight line in the plane, there exists an unending class A geodesic of the type of this straight line. Of these, only those of the type of periodic straight lines can be periodic. Since the number of non-periodic straight lines through a given point is non-denumerable, there must exist a non-denumerable infinity of class A non-periodic unending geodesics, no one of the type of a periodic straight line.

Since the phase-space is closed, a theorem[4] of Birkhoff holds to the effect that every geodesic contains among its limit geodesics a minimal set of geodesics and the number of geodesics in this minimal set is non-denumerable except in the simple case when the minimal set is a single periodic geodesic. A further result needed has been proved by Morse,[5] namely, that a geodesic which is a limit geodesic of class A geodesics is itself of class A. Furthermore, it is easy to prove that an infinite set of class A geodesics, all of the type of the same straight line, and such that the distance of any point of any member of the set from the given line does not exceed a fixed constant, can have as limit geodesics only geodesics of the type of that line.

Now consider a class A geodesic, g, of the type of a non-periodic straight line m. It can be assumed that g passes through the fundamental region. Let each of the congruent regions through which g passes be translated to the fundamental region. There results an infinite set, G, of geodesics, each representing g, but in the plane, no one coinciding with another. The limit geodesics of g are the limit geodesics of G. It can be proved that the distance which these geodesics wander from m is uniformly bounded. Applying the theorems stated in the preceding paragraph, it follows that among the limit geodesics of the set G, there is a minimal set of geodesics of the type of m. Since m is non-periodic, any member of this set is non-periodic and the number in the set is non-denumerable. Each geodesic of this non-denumerable set is a class A recurrent geodesic. This result, applied to a surface of genus one, becomes the

THEOREM. *Given a surface homeomorphic with a torus, the functions defining the correspondence being of class C^3, there exists on this surface a non-denumerable set of minimal sets of geodesics, each of these minimal sets containing a non-denumerable infinity of non-periodic recurrent geodesics.*

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THE GENERAL CASE OF INTEGRO-Q-DIFFERENCE EQUATIONS

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Integro-q-difference equations of the type

$$\sum_{i=0}^{n} a_i(x)y(q^{n-i}x) = b(x) + \int_0^x K(x, \xi)y(\xi)d\xi \quad (n \geqq 1), \qquad (1)$$

$$a_i(x) = a_0^{\,i} + a_1^{\,i}x + \ldots \quad (i = 0, 1 \ldots n),$$

$$b(x) = b_0 + b_1x + \ldots,$$

$$K(x, \xi) = K_{0,0} + K_{1,0}x + K_{0,1}\xi + \ldots,$$

$[b(x), a_0(x), a_n(x) \not\equiv 0$; the $a_i(x)$, $b(x)$ analytic at $x = 0$;
$K(x, \xi)$, analytic at $(x = 0, \xi = 0)]$

have been considered by several writers. For the complex plane an investigation was made by C. R. Adams.[1] Such a study necessarily depends on the nature of the characteristic equation

$$E_n(\zeta) = \sum_{i=0}^{n} a_0^i \zeta^{n-i} = 0, \tag{2}$$

(and of other characteristic equations, if any) and is based on the existence of solutions, in the complex plane, of the corresponding linear q-difference equation

$$\sum a_i(x)y(q^{n-i}x) = 0. \tag{3}$$

Now, an existence theorem for an equation of type (3) has been established by R. D. Carmichael[2] and its proof simplified by Adams;[3] both writers have restricted themselves, inasmuch as the analytic theory is concerned, to the case when the characteristic equation (2) has no zero or infinite roots. Necessarily in (A) the equation had to be studied for this case of the roots of (2).

My purpose is to obtain existence results for the equation (1) when several characteristic equations are admitted. In a paper on equations of type (3)[4] I have recently obtained certain existence results. The results in (T) are for the unrestricted case of the roots of the characteristic equation and they are as follows:

Let $|q| > 1 \quad (q = |q| e^{\sqrt{-1}\,\bar{q}};\ 0 \leqq \bar{q} < 2\pi)$;

then by a transformation

$$x = e^{t \log q} \quad (t = u + \sqrt{-1}\, v) \tag{4}$$

(3) assumes the form of an ordinary difference equation:

$$M_n(t; z(t)) \equiv M_n(z(t)) \equiv \sum \alpha_i(t) z(t + n - i) = 0. \tag{5}$$

Let W denote the part of the t-plane bounded above by a portion of a line with the slope $a_2[=(-\bar{q} - |\log q|)/\log |q| < 0]$ and bounded below by a portion of a line with the slope $a_1[=(-\bar{q} + |\log q|)/\log |q| > 0]$. A part of W on and above a line below the axis of reals and parallel to this axis, is denoted by $W(u)$. Similarly defined and extending downward is a region $W(l)$. Furthermore, $W^*(u)$ is to denote a subregion of $W(u)$ with the right boundary, for $|t|$ sufficiently great, consisting of a portion of a line with the slope $a_2 + \epsilon(\epsilon > 0)$. There is an analogous subregion $W^*(l)$ of $W(l)$. Finally, W^* is to be a subregion of the type $W^*(u) + W^*(l)$. The upper and lower boundaries of W certainly extend indefinitely upward and downward from the negative axis.[5] *The Fundamental Existence Theorem (of (T)) asserts that there exists a linearly independent set of solutions of* (5),

$$z_j(t) = e^{Q_j(t)} e^{\gamma_j t \log q} \sum_{H=0}^{K_j} t^H \eta_j^H(t), \tag{5a}$$

$$Q_j(t) = M_j t^s \log q \quad (M_1 \leqq M_2 \leqq \ldots \leqq M_n;\ j = 1 \ldots n)$$

where the $\eta_j^H(t)$ *are analytic in* W *(and in a more extensive region); moreover,*

$$\eta_j^H(t) \backsim \sigma_j^H(t) = \sigma_{j;0}^H + \sigma_{j;1}^H e^{\frac{t \log q}{s}} + \ldots \tag{5b}$$

$$(s = s_j;\ H = 0 \ldots k_j;\ j = 1 \ldots n)$$

for t *in a region* W^*. *The Preliminary Lemma (of* (T)*) asserts that there exists a set of solutions of* (5), *given by* (5a) *where the* $\eta_j^H(t)$ *are analytic and bounded in a region* $W(u)$, *while* (5b) *holds in a region* $W^*(u)$. *Similar results hold in* $W(l)$ *and* $W^*(l)$.

A formal series (which may be divergent) of the form

$$\sum_{H=0}^{K_j} t^H \sigma_j^H(t)$$

will be called a σ-series (as in (T)). The expressions

$$e^{Q_j(t)} e^{\gamma_j t \log q} \sum_{H=0}^{K_j} t^H \sigma_j^H(t)$$

are formal series solutions of (A). It will be assumed, as can be without loss of generality, that $\alpha_0(t) \equiv 1$.

Let U denote the t-half plane bounded on the right by a line L with the slope $\log|q|\bar{q}$. Let U^* be a subregion of U which, for $|t|$ sufficiently large, is bounded by portions of two lines extending upward and downward and making arbitrarily small angles with the corresponding extremities of L. More generally, U^* could be defined, for instance, as a subregion of U which, sufficiently far from the origin, is bounded by a curve B^u (extending upward) and a curve B^l (extending downward) such that

$$t = t' - c|It|^{\epsilon}$$

$$[c > 0,\ \epsilon > 0;\ t \text{ on } B^u (\text{or } B^l);\ t' \text{ on } L],$$

whenever $It = It'$. Here the constant ϵ could be taken arbitrarily small.

For the purpose at hand the following special existence result will be needed.

LEMMA. *Suppose that the coefficients* $\alpha_i(t)$ *of* (5) *are analytic in* U^* *and that in* U^* *they satisfy the asymptotic relations*

$$\alpha_i(t) \sim a_0^i + a_1^i e^{\frac{t \log q}{s}} + \ldots \tag{5c}$$

Consider all the formal series solutions belonging to the smallest M of (5). *To each such series there will correspond a solution of* (5) *asymptotic to it in U^* and analytic in U^*.*

To prove this we need only to form the corresponding difference system

$$Y(t+1) = D(t)Y(t); \quad Y(t) \equiv (y_{ij}^{(t)}),$$

$$D(t) = \begin{pmatrix} 0, 1 \ldots 0 \\ 0, 0, 1 \ldots 0 \\ \ldots \quad \ldots \quad \ldots \\ -\alpha_n, \ldots \ldots -\alpha_1 \end{pmatrix}$$

and to apply the method of iterations[6] as in (*BT*; §3) on the basis of the formal matrix solution of the system. The process of iterations will converge, yielding the required result, for each column which corresponds to the smallest M.

By (4) the equation (1) becomes

$$M_n(z(t)) = \beta(t) + \int_t^\infty W(t, r)z(r)d(e^{r \log q}), \tag{6}$$

$$\beta(t) = b(x), \ \alpha_i(t) = a_i(x) \quad (i = 0, 1 \ldots n),$$

$$W(t, r) = -K(e^{t \log q}, e^{r \log q}).$$

A solution will be found by successive approximations on the basis of the set of equations (the same as in (*A*))

$$M_n(z(t)) = 0, \tag{7}$$

$$M_n(z^{(1)}(t)) = \beta(t), \tag{7a}$$

$$M_n(z^{(m)}(t)) = \beta(t) + \int_t^\infty W(t, r)z^{(m-1)}(r)d(e^{r \log q}) \equiv \beta^{(m-1)}(t) \quad (m = 2, 3 \ldots). \tag{7b}$$

Now the difference equation of order $(n+1)$,

$$M^{(1)}{}_{n+1}(t; z(t)) \equiv \beta(t)M_n(t+1; z(t+1)) - \beta(t+1)M_n(t; z(t)) = 0, \tag{8}$$

is related to (7) as follows. If $z(t)$ is a solution of (8) the function $p(t)$,

$$p(t) = \frac{M_n(t; z(t))}{\beta(t)}, \tag{8a}$$

will be of period unity. Every solution of (7) and (7a) will be a solution of (8). The characteristic equation of (8) which corresponds to the smallest M of (8), that is to $M = 0$, will have for roots $\zeta = q^m (b_0 = \ldots = b_{m-1} = 0; b_m \neq 0)$ and the non-zero roots of the equation (2). In general, of course, there will be also some other characteristic equations related to (8). The

equation (8) will be satisfied by a set of n formal series solutions of (7). There will also be a formal solution of (8),

$$\sigma(t) = e^{m\, t \log q} \sum_{H=0}^{K} t^H \sigma_{n+1}^{H}(t) (= \sigma\text{-series};\ s = 1), \tag{9}$$

which will correspond to the root $\zeta = q^m$ and which will be a formal solution of (7a). Assume that the coefficients in $M_n(z(t))$ are of type (5c) and also that

$$\beta(t) \sim b_0 + b_1 e^{t \log q} + \ldots \tag{9a}$$

in U^s. There exists then a solution of (8) of the form

$$z^{(1)}(t) = \sum_{H=0}^{K} t^H \eta^{(1)H}(t) \tag{10}$$

where the $\eta^{(1)H}(t)$ are analytic (for $t \neq \infty$) in a region U^s, while

$$z^{(1)}(t) \sim \sigma(t) \tag{10a}$$

in U^s. In virtue of (10a), and since $\sigma(t)$ is a formal solution of (7a),

$$M_n(t;\ z^{(1)}(t)) \sim \beta(t) \quad (\text{in } U^s)$$

so that, in particular, in a strip along the negative axis

$$p(t) \sim 1.$$

Since $p(t)$ is analytic and periodic it follows that $p(t) \equiv 1$. Hence (10) is a solution of (7a). *Thus, generally speaking, we have determined the nature of an analytic solution of the non-homogeneous linear difference equation* (7a). The relation of this result to the corresponding q-difference problem is obvious.

Assume now that the characteristic equation (2) (the one corresponding to $M = 0$) has no root of the form q^p ($p \geqq 0$; p, integer). No restriction is made concerning the roots of other characteristic equations (if there are any). Then there exists a solution, (10), of (7a) with $k = 0$. The function $\beta^{(1)}(t)$, occurring in (7b) for $m = 2$, will be analytic in U^s. Moreover, in U^s

$$\beta^{(1)}(t) \sim \beta_0^{(1)} + \beta_1^{(1)} e^{t \log q} + \ldots$$

$$(\beta_0^{(1)} = \ldots = \beta_{m^{(1)}-1}^{(1)} = 0;\ \beta_{m^{(1)}}^{(1)} \neq 0;\ m^{(1)} \geqq 0).$$

There is an arbitrariness of $\sigma(t)$ enabling a choice of $\sigma(t)$ for which not all the $\beta_i^{(1)}$ ($i = 0, 1, 2 \ldots$) will be zero. The equation of order $(n+1)$,

$$M_{n+1}^{(2)} t;\ z(t) \equiv \beta^{(1)}(t) M_n(t+1;\ z(t+1)) - \beta^{(1)}(t+1) M_n(t;\ z(t)) = 0, \tag{11}$$

will have $M = 0$ as the smallest M. Its characteristic equation, corresponding to $M = 0$, will be

$$\left(\zeta - q^{m^{(1)}}\right)E_n(\zeta) = 0.$$

There will be a formal solution of (11), belonging to $\zeta = q^{m^{(1)}}$,

$$\sigma^{(1)}(t) = e^{m^{(1)}t^{\log q}} \sum_{H=0}^{K^{(1)}} t^H \sigma_{n+1}^{(1)H}(t) \quad (= \sigma\text{-series};\ s = 1) \qquad (11a)$$

which will also be a formal solution of (7b; $m = 2$). As no roots of $E_n(\zeta) = 0$ are of the form q^p (integer $p \geqq 0$), such a solution (11a) clearly exists with $k^{(1)} = 0$. The analytic solution $z^{(2)}(t)$ of (11), corresponding to (11a) will be analytic in U^ϵ, while

$$z^{(2)}(t) \sim \sigma^{(1)}(t) \qquad (11b)$$

in a region U^ϵ. In virtue of (8a) (with $\beta(t)$ replaced by $\beta^{(1)}(t)$) and by (11b) it follows that $z^{(2)}(t)$ is a solution of (7b; $m = 2$).

By induction the following can be shown. The equation (7b; $m = v$) is satisfied by a solution $z^{(v)}(t)$, analytic in a region U^ϵ, such that

$$z^{(v)}(t) \sim \sigma^{(v-1)}(t) \quad (= \sigma\text{-series};\ s = 1) \qquad (12)$$

in U^ϵ. Moreover, if L is sufficiently far to the left, by induction it follows that

$$\lim_{v} z^{(v)}(t) = z^*(t) \qquad (12a)$$

uniformly for t in U^ϵ. The function $z^*(t)$ will be an analytic solution of (6).

THEOREM. *Let* $|q| > 1$. *Assume* (5c) *and* (9a). *Then the equation* (6) (*related to* (1) *and with* $\alpha_0(t) \equiv 1$) *has a solution* $z^*(t)$, *analytic in* U^ϵ *and defined by* (12) *and* (12a), *provided that the characteristic equation* (2) (*corresponding to the smallest* M) *has no roots of the form* q^p (p, *a non-negative integer*). *A solution, analytic in* U, *will certainly exist if the* $\alpha_i(t)$ *and* $\beta(t)$ *are representable by convergent series.*

It is not likely that, in general, analytic solutions exist when the characteristic equation corresponding to $M = 0$ has roots of the form q^p (p, non-negative integer). The condition on the roots introduced in the above theorem is less restrictive than the corresponding one which the method of (*A*) would require. Besides, in (*A*) a single characteristic equation is assumed.

[1] C. R. Adams, "Note on Integro-q-Difference Equations," *Trans. Am. Math. Soc.*, **31**, No. 4, 861–867, hereafter referred to as (*A*). References to other writers are found in (*A*).

[2] R. D. Carmichael, "The General Theory of Linear Difference Equations," *Am. J. Math.*, **34**, 147–168 (1912).

[3] Adams, "On the Linear Ordinary q-Difference Equation," *Annals of Math.*, **30**, 195–205 (1929). In this paper the nature of the formal series solutions of (3), for the general case, is also determined.

[4] W. J. Trjitzinsky, "Analytic Theory of Linear q-Difference Equations," hereafter referred to as (T). This paper will appear in the *Acta Mathematica*.

[5] For $W^e(u)$, $W^e(l)$, W^e more general subregions of $W(u)$, $W(l)$, W, respectively, can be taken (cf. (*T*; § 2)).

[6] Concerning iterations cf. G. D. Birkhoff, "General Theory of Linear Difference Equations," *Trans. Am. Math. Soc.*, **12**, 243–284 (1911) and Birkhoff and Trjitzinsky, "Analytic Theory of Singular Difference Equations," *Acta Mathematica*, **60**, 1–2, 1–89 (1932). The latter paper will be referred to as (*BT*).

A PROPERTY OF INDEFINITELY DIFFERENTIABLE CLASSES

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A convenient way of defining classes of functions $f(x)$, indefinitely differentiable on an interval (a, b) $[a < b]$, is by letting C_A denote the class of functions such that

$$|f^{(v)}(x)| < f^v A_v \quad (v = 0, 1\ldots;\ a \leqq x \leqq b), \tag{1}$$

where the sequence (A_v) is independent of any particular member of the class. However, f may be a different constant for various functions of of C_A.[1] A class C_A is additive; that is, if $a(x)$ and $b(x)$ belong to C_A the function $c(x)(= a(x) + b(x))$ will be also of C_A; the same will be true of $a(x) - b(x)$. Some classes C_A are analytic. Under certain conditions on (A_v) a class C_A will be quasi-analytic (cf. (C)).[2] There exist quasi-analytic classes C_A which are not analytic. A class C_A is quasi-analytic if and only if there exist no member of the class, $f(x)$, such that

$$f^{(v)}(x_0) = 0 \quad (v = 0, 1, 2\ldots;\ x_0, \text{ a fixed point in } (a, b)), \text{ while}$$

$$f(x) \not\equiv 0.$$

Quasi-analytic classes thus form a natural and an important extension of analytic classes. The members of a quasi-analytic class C_A are determined uniquely throughout the interval by the values of the derivatives, of all orders,[3] at a *single* point of (a, b). It is therefore of importance to settle the following question.

Do there exist classes C_A, distinct from quasi-analytic classes, specified by the property that the members of C_A are determined uniquely throughout (a, b) by the values of the derivatives, of all orders, at $n(\geqq 2)$ distinct points of (a, b)?

The answer, in the negative, will be formulated in the theorem

THEOREM. *There exist no classes C_A, distinct from quasi-analytic classes,*

specified by the property that the members of C_A are determined uniquely throughout (a, b) by the values of the derivatives, of all orders, at $n(\geqq 2)$ distinct points of (a, b).

In fact, suppose the theorem is not true. Then there exists a non-quasi-analytic class C_A with the properties whose possibility (for non-quasi-analytic classes) has been denied in the theorem. Let the x_i $(i = 1, 2 \ldots n;\ a \leqq x_i \leqq b)$ be a set of points referred to above. There exists a function $f(x)$,

$$|f^{(v)}(x)| < f^v A_v \quad (v = 0, 1 \ldots;\ a \leqq x \leqq b) \tag{2}$$

such that $f^{(v)}(x_0) = 0$ $(v = 0, 1 \ldots)$ for a fixed x_0 in (a, b), while $f(x) \not\equiv 0$. Suppose, as is possible without any loss of generality, that $a = 0$ and $b = 1$. Let $p(x)$ be a polynomial, with real coefficients and different from a constant, such that

$$p(x_i) = 0 \quad (i = 1, 2 \ldots n). \tag{3}$$

Form the function

$$\varphi(x) = \sin^2 (p(x) + b), \tag{4}$$

where $\sin^2 b = x_0$. Then

$$\begin{aligned} 0 \leqq \varphi(x) \leqq 1 \quad & (0 \leqq x \leqq 1), \\ \varphi(x_i) = x_0 \quad & (i = 1, 2 \ldots n). \end{aligned} \tag{4a}$$

It is known that, if $a(x)$ is analytic on $(0, 1)$,

$$|a^{(v)}(x)| < (av)^v \quad (v = 0, 1 \ldots;\ \ 0 \leqq x \leqq 1)$$

where a is a constant depending on $a(x)$.[4] Accordingly,

$$|\varphi^{(v)}(x)| \leqq (\varphi v)^v \quad (v = 0, 1 \ldots;\ 0 \leqq x \leqq 1)$$

and, for a suitable φ_1,

$$|\varphi^{(v)}(x)| \leqq \varphi_1^v v! \quad (v = 0, 1 \ldots;\ \ 0 \leqq x \leqq 1). \tag{4b}$$

Let M_v be the upper bound of $|f^{(v)}(x)|$ for $0 \leqq x \leqq 1$. On noting that $f^{(v)}(x_0) = 0$ $(v = 0, 1 \ldots)$, it is concluded that

$$f^{(v)}(z) = \frac{1}{(m - v - 1)!} \int_{x_0}^{z} (z - s)^{m - v - 1} f^{(m)}(s) ds \quad (v \leqq m).$$

Hence it is observed that

$$|f^{(v)}(z)| \leqq M_v \leqq \frac{M_m}{(m - v)!} \quad (v \leqq m;\ \ 0 \leqq z \leqq 1) \tag{5}$$

(compare with (*C;* p. 22)). The following fact, stated in (*C;* p. 22), will be now noted (with modified notation).

Suppose that

$$a \leqq \varphi(x) \leqq b \qquad (\text{for } x_0' \leqq x \leqq x_1),$$
$$\left| f^{(v)}(z) \right| \leqq B_v \qquad (\text{for } a \leqq z \leqq b;\ v = 0, \ldots m),$$
$$\left| \varphi^{(v)}(x) \right| \leqq C_v \qquad (\text{for } x_0' \leqq x \leqq x_1;\ v = 0, 1 \ldots m).$$

Let

$$f_m^*(z) = \left(\sum_{v=0}^{m} \frac{B_v}{v!} z^v \right) + \ldots,$$

$$\varphi_m^*(x) = \left(\sum_{v=1}^{m} \frac{C_v}{v!} x^v \right) + \ldots.$$

Then

$$\left| \frac{d^m f(\varphi(x))}{dx^m} \right| \leqq \left\{ \frac{d^m}{dx^m} f_m^*[\varphi_m^*(x)] \right\}_{x=0} \quad (x_0' \leqq x \leqq x_1).$$

Consider the function $g(x) = f(\varphi(x))$. In the above-mentioned device we let $a = x_0' = 0$, $b = x_1 = 1$, $B_v = \dfrac{M_m}{(m-v)!}$ $(v = 0, 1 \ldots m;$ cf. (5)) and $C_v = \varphi_1{}^v v!$ $(v = 0, 1 \ldots m;$ cf. (4b)). Thus, for $f_m^*(z)$ and $\varphi_m^*(x)$ the following functions may be chosen

$$f_m^*(z) = \sum_{v=0}^{m} \frac{M_m}{(m-v)!v!} z^v = \frac{M_m}{m!} (1+z)^m,$$

$$\varphi_m^*(x) = \sum_{v=1}^{\infty} \varphi_1{}^v x^v = \frac{\varphi_1 x}{1 - \varphi_1 x} \left(|x| < \frac{1}{\varphi_1} \right).$$

Then

$$f_m^*[\varphi_m^*(x)] = \frac{M_m}{m!} (1 - \varphi_1 x)^{-m}.$$

Consequently, for $0 \leqq x \leqq 1$,

$$\left| g^{(m)}(x) \right| \leqq \frac{M_m}{m!} \left\{ \frac{\varphi_1^m m(m+1)\ldots(2m-1)}{(1-\varphi_1 x)^{2m}} \right\}_{x=0}$$

$$= \varphi_1^m M_m \frac{m \ldots (2m-1)}{m!} < (2e\varphi_1)^m M_m < (2e\varphi_1 f)^m A_m \ (m = 0, 1, 2 \ldots).$$

Here the constant $(2e\varphi_1 f)$ is independent of the choice of the set of points $(x_1, x_2 \ldots x_n)$.

Accordingly, $g(x)$ is of the class C_A. Since $f(x) \not\equiv 0$, it follows that $g(x) \not\equiv 0$. Moreover, in virtue of (4a) and since $f^{(v)}(x_0) = 0$ $(v = 0, 1 \ldots)$, we conclude that

$$g^{(v)}(x_i) = 0 \quad (i = 1, 2 \ldots n;\ v = 0, 1, 2 \ldots).$$

However, existence of such a function in the class C_A means, in virtue of the additive property of C_A, that the members of this class are not uniquely determined throughout the interval of definition by the values of the

derivatives of all orders at $n(\geqq 2)$ distinct points of the interval. This completes the proof of the theorem.

In a different formulation this theorem can be stated as follows.

For quasi-analiticity of C_A (in (a,b)) it is necessary and sufficient that the functions of C_A should be determined uniquely (in (a,b)) by the values of the derivatives of all orders at $n(\geqq 2)$ distinct points (of (a,b)).

[1] This is the definition used by T. Carleman in his book *Les fonctions quasi analytique*, Paris, 1926, hereafter referred to as (*C*).

[2] Also see É. Borel, *Leçons sur les fonctions monogènes*, Paris, 1917, and W. J. Trjitzinsky's papers on quasi-analytic functions in the *Annals of Math.*, **30**, 526–546 (1929); **32**, 623–658 (1931); **32**, 659–685 (1931).

[3] The derivative of order 0, $f_{(x)}^{(0)}$, of $f(x)$ is $f(x)$.

[4] Cf., for instance, C. de la Vallée Poussin, "On the Approximation. . .," *The Rice Institute Pamphlet*, Vol. XII, April, 147 (1925).

OPERATORS OF ORDER p^m IN THE GROUP OF ISOMORPHISMS OF THE ABELIAN GROUP OF ORDER p^n AND TYPE 1, 1, . . .

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We propose to characterize the operators in question in terms of the well-known invariants of linear transformations. We take a group H of order p^n and type 1, 1, . . . and represent one of its automorphisms as a non-homogeneous linear transformation on the exponents of a set of independent generators. It has been shown in a recent paper[1] that if the isomorphism U of order p transforms one of the operators of the abelian group H_k of order p^k and type 1, 1, . . . into a set of generators of H_k, then U may be written in the form

$$(1) \qquad U = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_k \end{pmatrix}$$

where $a_j = -\binom{r+1+k-j}{r}$, $r = p-1-k \geqq 0$. Dickson shows[2] that every linear transformation on n variables may be written in a canonical form which is a set of transformations of the type (1) on distinct *chains* of variables. The variables of a chain are linearly independent, and no set of variables from one chain is linearly dependent on

variables from other chains. The transformation (1) is called a partial transformation. The length of any chain of variables determined by a given transformation will depend in general on the variable chosen as the leader, but there will be a maximum length for the transformation such that no leader gives rise to a chain of greater length. Let the first chain be of maximum length, and let each chain thereafter be selected so that its length is as great as possible, its leader being linearly independent of the variables appearing in the chains determined up to that point. Then the characteristic determinants of the partial transformations on the separate chains are, apart from signs, the invariant factors of the transformation; each is divisible by the one which follows it. Now the characteristic determinant of the partial transformation (1), in which $k < p$, is

$$(-1)^k \ (\lambda^k - a_k\lambda^{k-1} - a_{k-1}\lambda^{k-2} - \ \ldots \ - a_1).$$

If in this we substitute the values of $a_j, j = 1, 2, \ldots k$, a short computation will show that the characteristic determinant of (1) is

$$\Delta(\lambda) \equiv (-1)^k \ (\lambda - 1)^k, \text{ mod } p. \tag{2}$$

The partial transformation will take the form (1) even if $k \not< p$, in which case $k = p$ if the order of the transformation is p, since the conjugates of an operator of H under the given transformation can generate a group of order at most p^p. In that case we have $a_1 = 1$, $a_2 = a_3 = \ldots = a_K = 0$, and the characteristic determinant is $\Delta(\lambda) = (-1)^K (\lambda^K - 1)$ which is congruent to (2), mod p.

Conversely, if the characteristic determinant of the transformation is $(-1)^n \ (\lambda - 1)^n$, and if, moreover, none of the chains is of length greater than p, then the partial transformations are all of order p, and U is of order p. Hence we have

(3) *Necessary and sufficient conditions that an operator U in the group of isomorphisms of the abelian group of order p^n and type 1, 1, ... be of order p are that*

(1) *the characteristic determinant of U be $(-1)^n \ (\lambda - 1)^n$, and*

(2) *U have no invariant factor of degree greater than p.*

It was shown in the paper first referred to[3] that the operators in the group of order p determined by the partial transformation (1) which are invariant under (1) constitute a group of order p. It is likewise true and easily verified in case $k = p$. Hence, each partial transformation determines such a subgroup and every operator of H which is invariant under U is in the group generated by subgroups determined in the above manner by the respective partial transformations. Since each partial transformation determines an invariant factor distinct from unity, we have

(4) *The subgroup of H composed of operators invariant under U con-*

stitute a group of order p^δ, where δ is the number of invariant factors of U distinct from unity.

The theorems (3) and (4), which follow from theorems in the paper referred to, may be generalized immediately by going back to the methods of that paper and removing the restriction in (1) that k be less than p. The argument goes through without change excepting that in some obvious places p is replaced by a power of p. There are three points essential to the argument. If the order of U is p^m then (*a*) H_k of order p^k contains but one subgroup of order p whose operators are invariant under U, (*b*) the a's for each partial transformation are determined and are such that the characteristic determinant is (2), and (*c*) the group of isomorphisms of H_k where $k = p^m - 1$, contains an operator of order p^m whose canonical form consists of a single chain. From these facts we may draw the following conclusions:

(5) *A necessary and sufficient condition that an operator U in the group of isomorphisms of the abelian group of order p^n and type 1, 1, ... be of order a power of p, is that its characteristic determinant be $(-1)^n(\lambda - 1)^n$.*

(6) *If U is of order p^m and α is the degree of the invariant factor of highest degree of U, then $p^{m-1} < \alpha \leq p^m$; and, conversely, if α is any number such that $p^{m-1} < \alpha \leq p^m$; then the order of U is p^m.*

Moreover, (4) as stated for U of order p holds for U of order p^m.

[1] "Prime-Power Abelian Groups Generated by a Set of Conjugates under a Special Automorphism." § 4. This paper has been submitted to the Amer. Journal of Math.

[2] "Modern Algebraic Theories," p. 90, Theorem I. See footnote, p. 89, for other references.

[3] Loc. cit. (4.6).

A CHARACTERIZATION OF FIELDS IN THE CALCULUS OF VARIATIONS

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1. *Introduction.*—The present paper is concerned with a geometric characterization of focal points and conjugate points in the calculus of variations. In the plane it is easy to prove that conjugate points of a point P on an extremal g are the points on g whose neighborhoods are not covered in a one-to-one manner by the family of extremals issuing from R with directions near that of g at P. In fact one can show that the extremals of this family intersect g arbitrarily near the conjugate points of P if the directions at P are sufficiently near those of g.

No such method suffices in n-space for $n > 2$. The difficulty centers around the fact that we are concerned with properties of families of extremals, which do not necessarily hold for families of curves which are not extremals, as we show by example in § 3. In spite of these difficulties we shall show that in the analytic case focal points and conjugate points can be characterized by the failure of certain families of extremals to cover the neighborhoods of these points in a one-to-one manner.

2. *Hypotheses.*—Let x and $(y_1, \ldots, y_n)$ be rectangular coördinates in an Euclidian $n + 1$ space. Let R be an open region in this space. Let

$$f(x, y_1, \ldots, y_n\ p_1, \ldots, p_n) = f(x, y, p)$$

be a function analytic in its arguments for (x, y) on R and (p) any set of numbers. Our integral shall have the form

$$J = \int_a^b f(x, y, y')dx.$$

In the region R let there be given an extremal g of the form $y_i = \bar{y}_i(x)$ where the functions $\bar{y}_i(x)$ are analytic for x on an interval $a \leqslant x \leqslant b$. We assume that[2]

$$f_{p_ip_j}[x, \bar{y}(x), \bar{y}'(x)]z_iz_j > 0 \quad (i, j = 1, 2, \ldots, n)$$

for x on the interval $a \leqslant x \leqslant b$ and for any set of numbers $(z) \neq (0)$.

3. *Focal Points.*—Let Σ be any regular analytic manifold[3] cutting g at a point A and not tangent to g at A. We suppose that Σ cuts the extremal g transversally at A. There will then exist a family of extremals neighboring g cut transversally by Σ at points (α) on Σ near A, and representable in the form

$$y_i = y_i(x, \alpha) \quad (i = 1, 2, \ldots, n) \tag{3.1}$$

where the functions $y_i(x, \alpha)$ are analytic in x and (α), for $a \leqslant x \leqslant b$ and (α) near (0).[4] The Jacobian

$$\Delta(x) = \frac{D(y_1, \ldots, y_n)}{D(\alpha_1, \ldots, \alpha_n)} \quad (\alpha) = (0)$$

does not vanish at A on g. The points on g at which $\Delta(x) = 0$ are called the *focal points* of Σ on g. That the focal points are isolated follows from a note cited below.[5]

Let (x_0, y_0) be a particular point on g. We shall say that the family of extremals (3.1) *simply covers* the neighborhood of the point (x_0, y_0) if the points (x, y) determined in (3.1) by (x, α), for x sufficiently near x_0 and (α) sufficiently near (0), are in one-to-one correspondence with the sets (x, α).

As we stated above, it is not true for an n-parameter family of curves (represented by functions sufficiently differentiable) neighboring a pre-

ferred curve g, that the isolated vanishing of the Jacobian of the family of curves at a point P on g implies that the family fails to cover the neighborhood of P in a one-to-one manner. In fact the following is a simple example to the contrary. In the plane let

$$y = (x - \alpha)^3. \tag{3.2}$$

Through each given point (x, y) there passes a unique curve of the family (3.2).

On the other hand consider the family (3.2) neighboring any preferred curve g, obtained by setting $\alpha = \alpha_0$. The Jacobian of (3.2) evaluated along g is

$$\Delta(x) = \left.\frac{\partial y}{\partial \alpha}\right|_{\alpha = \alpha_0} = -3(x - \alpha_0)^2.$$

This Jacobian is not identically zero, but vanishes at $x = \alpha_0$. Hence it is possible to have families of curves (but not extremals as we shall see) which cover the neighborhood of a focal point in a one-to-one manner.

Our geometric characterization of focal points is given by the following theorem:

THEOREM 1. *A necessary and sufficient condition that a point (x_0, y_0) on g be a focal point of Σ is that the family of extremals cut transversally by Σ near g shall fail to cover the neighborhood of (x_0, y_0) simply.*

To state that the condition is sufficient is merely to say it implies the vanishing of the Jacobian $\Delta(x)$ at $x = x_0$, a well-known fact.

We now let $\Delta(x_0) = 0$ and seek to prove that the family of extremals (3.1) fails to cover the neighborhood of (x_0, y_0) simply. To begin with, we take x_0 as the first focal point of Σ on g following Σ. The general case will follow readily.

(A) *We shall assume that the family of extremals* (3.1) *covers the neighborhood of (x_0, y_0) simply.*

Using this assumption we shall arrive at a contradiction of the well-known fact that if $\Delta(x_0) = 0$ no segment of g extending from the point (x_1, y_1), the intersection of g and Σ, to a point (x_2, y_2) on g beyond x_2 can give a weak minimum to J relative to neighboring analytic curves joining (x_2, y_2) to points on Σ near (x_1, y_1).

Under assumption (A) the family of extremals (3.1) simply covers a region S including a neighborhood of each point of g on the interval $x_1 \leqslant x \leqslant x_2$ provided that x_2 is sufficiently near x_0 and (α) is sufficiently near (0). We do not have what is commonly called a "field" in the calculus of variations because we do not have the additional hypothesis that the Jacobian $\Delta(x) \neq 0$ on the interval $x_1 \leqslant x \leqslant x_2$. Lacking this hypothesis our problem is to establish the existence of the weak minimum making essential use of (A) in its place.

The given correspondence F between the points (x, y) of S and the sets (x, α), established by (3.1) is one-to-one and continuous one way. As is well known[6] it is then continuous both ways.

The sets (α) corresponding under F to the points (x, y) on S will then be given by continuous functions $\alpha_i(x, y)$. The extremal of the family (3.1) through the point (x, y) has the slope functions

$$p_i(x, y) = y_{i_x}[x, \alpha(x, y)] \quad (i = 1, 2, \ldots, n),$$

where $p_i(x, y)$ is continuous in (x, y) for (x, y) in S. (If we were dealing with a proper field, the functions $p_i(x, y)$ would be at least of class C'' and classical proofs make essential use of this fact).

We now consider the Hilbert integral

$$I = \int [f(x, y, p) - p_i f_{p_i}(x, y, p)]dx + f_{p_i}(x, y, p)dy_i \qquad (i = 1, 2, \ldots, n)$$

where the arguments (x, y, p), are $[x, y, p(x, y)]$. We cannot apply the usual condition that the integral be independent of path because the coefficients of dx and dy_i are not necessarily differentiable.

Along an extremal of the family (3.1), however, we have

$$dy_i = y_{i_x}(x, \alpha)dx = p_i(x, y)dx,$$

from which the following well-known lemma results.

LEMMA 1. *Along an extremal of the family* (3.1) *the Hilbert integral* I *equals the integral* J.

In the space (x, α) we now consider the line integral

$$I^* = \int A(x, \alpha)dx + B_i(x, \alpha)d\alpha_i \quad (i = 1, 2, \ldots, n)$$

obtained from the integrand of I by setting

$$\begin{aligned} Adx + B_i d\alpha_i &= (f - f_{p_j}p_j)dx + f_{p_j}(y_{i_x}dx + y_{j\alpha_i}d\alpha_i) \\ &= fdx + y_{j\alpha_i}f_{p_j}d\alpha_i \end{aligned} \tag{3.3}$$

where the arguments on the right, (x, y, p), are

$$[x, y(x, \alpha), y_x(x, \alpha)]$$

If a curve γ^* of the form

$$\begin{cases} x = x(t) & t_1 \leqslant t \leqslant t_2 \\ \alpha_i = \alpha_i(t) & (i = 1, 2, \ldots, n) \end{cases} \tag{3.4}$$

is given in the (x, α) space with the functions (3.4) of class C', and if γ is the corresponding curve in the (x, y) space it is seen that

$$I^*_{\gamma^*} = I_\gamma.$$

On the other hand, if an admissible curve γ is first given in the (x, y)

space the corresponding curve γ^* in the (x, α) space may conceivably not even be rectifiable, so that $I^*_{\gamma^*}$ may conceivably not exist.

If, however, the functions $y_i(x, \alpha)$ are analytic, as in the present case, and if the curve γ is of the form

$$y_i = \phi_i(x) \qquad a \leqslant x \leqslant b \tag{3.5}$$

where the functions $\phi_i(x)$ are analytic in x, then it can be shown by a recourse to the theory of functions of several complex variables[7] that the real points (x, α) satisfying the condition

$$\phi_i(x) = y_i(x, \alpha) \tag{3.6}$$

neighboring a particular set (x^0, α^0) which satisfies (3.6) will be given for real values of a real parameter t by functions

$$\begin{aligned} x &= x(t) \qquad (i = 1,2, \ldots, n) \\ \alpha_i &= \alpha_i(t) \end{aligned} \tag{3.7}$$

which are real and analytic for t near 0. (That the real locus satisfying (3.6) depends upon one parameter t and not several is a consequence of the fact that the image γ^* of (3.5) in the (x, α) space is a curve.) Now the curve can be covered by a finite number of overlapping segments representable in the form (3.7). On each of these segments and the corresponding segments in the (x, y) space $I^* = I$. We are thus led to the following lemma:

LEMMA 2. *The image γ^* in the space (x, α), of an analytic curve γ of the form (3.5) is a curve along which*

$$I^*_{\gamma^*} = I_\gamma.$$

One can prove in essentially the usual manner that the integral I^* in the (x, α) space is independent of the path in a simply connected region. Sufficient conditions for this are that

$$\frac{\partial A}{\partial \alpha_i} \equiv \frac{\partial B_i}{\partial x} \tag{3.8}$$

$$\frac{\partial B_h}{\partial \alpha_j} \equiv \frac{\partial B_j}{\partial \alpha_h} \qquad (h, i, j = 1, 2, \ldots, n). \tag{3.9}$$

The conditions (3.8) are satisfied by virtue of the Euler equations. To show that the conditions (3.9) are satisfied in the present case we evaluate the integral J along any curve (α) of the family (3.1), from the point $x(\alpha)$ on Σ to a general point x, obtaining thereby a function

$$J(x, \alpha) = \int_{x(\alpha)}^{x} f(x, y(x, \alpha), y_x(x, \alpha))dx.$$

Upon using the fact that Σ cuts the family (3.1) transversally we find that

$$\frac{\partial J}{\partial \alpha_h} = f_{y_i} \frac{\partial y_i}{\partial \alpha_h}.$$

The conditions (3.9) are then a consequence of the fact that

$$\frac{\partial^2 J}{\partial \alpha_j \partial \alpha_h} = \frac{\partial^2 J}{\partial \alpha_h \partial \alpha_j}.$$

The integral I^* is accordingly independent of the path in the space (x, α). From lemma 2 the following conclusion is reached.

LEMMA 3. *The Hilbert integral*

$$I = \int (f - p_i f_{p_i}) dx + f_{p_i} dy_i,$$

with arguments

$$[x, y, p] = [x, y, p(x, y)]$$

is independent of path among paths of the form (3.5) *in the region S of the* (x, y) *space.*

Now, as in the classical theory, use of the Weierstrass E function leads to the next lemma.

LEMMA 4. *The extremal g yields a weak minimum to J relative to nearby curves of the form* (3.5) *joining* Σ *sufficiently near g to the point* (x_2, y_2) *on g where* $x_2 > x_0$.

If, however, $\Delta(x) = 0$ at $x = x_0$, according to the classical theory there will exist a nearby curve γ joining a point on Σ near g to (x_2, y_2) on g, and giving to J a value J_γ, such that $J_\gamma < J_g$. One can suppose, moreover, that γ is analytic, for in the contrary case one could approximate so closely to γ by an analytic curve $\bar{\gamma}$, joining γ's end-points, that $J_{\bar{\gamma}} < J_g$.

The conclusion in Lemma 4 was reached on the basis of assumption (A). This, however, is in contradiction to the facts just cited. Hence assumption (A) is false and the proof of Theorem 1 is complete if x_0 is the first focal point following Σ.

Consider any subsequent focal point. Without loss of generality, we can suppose that the integrand $f \neq 0$ along g. For in the contrary case, one could add a constant to f and bring this about. With this change we can now be sure of the existence of a regular analytic manifold Σ cutting the given family of extremals transversally relative to the new integrand and passing through a point of g separating P_0 from the preceding focal point or conjugate point. The given family of extremals will still be extremals and the Hilbert integral will still be independent of paths of the form (3.5) as before. We proceed now as before.

The proof of the theorem is thus complete.

[1] NATIONAL RESEARCH FELLOW when results of paper were obtained.

[2] A repeated subscript shall indicate summation with respect to that subscript.

[3] We are here concerned with a manifold on which the coördinates (x, y) of points near A are representable as functions analytic in n parameters (α), and such that not all of the Jacobians of the coördinates with respect to the parameters (α) are zero. We suppose that $(\alpha) = (0)$ corresponds to A.

[4] See Mason and Bliss, "The Properties of Curves in Space, Which Minimize a Definite Integral," *Trans. Amer. Math. Soc.*, **9**, 448 (1908).

[5] See Marston Morse, "The Order of Vanishing of a Conjugate Determinant," *Proc. Nat. Acad. Sci.*, **17**, 319 (1931).

[6] See V. Kerékjártó, *Vorlesungen uber Topologie*, I (Berlin), 34, 1923.

[7] See W. F. Osgood, *Lehrbuch der Funktionentheorie*, **2**, 1, Berlin (1929), and B. O. Koopman and A. B. Brown, "On the Covering of Analytic Loci by Complexes," *Trans. Amer. Math. Soc.*, **34**, 231 (1932).

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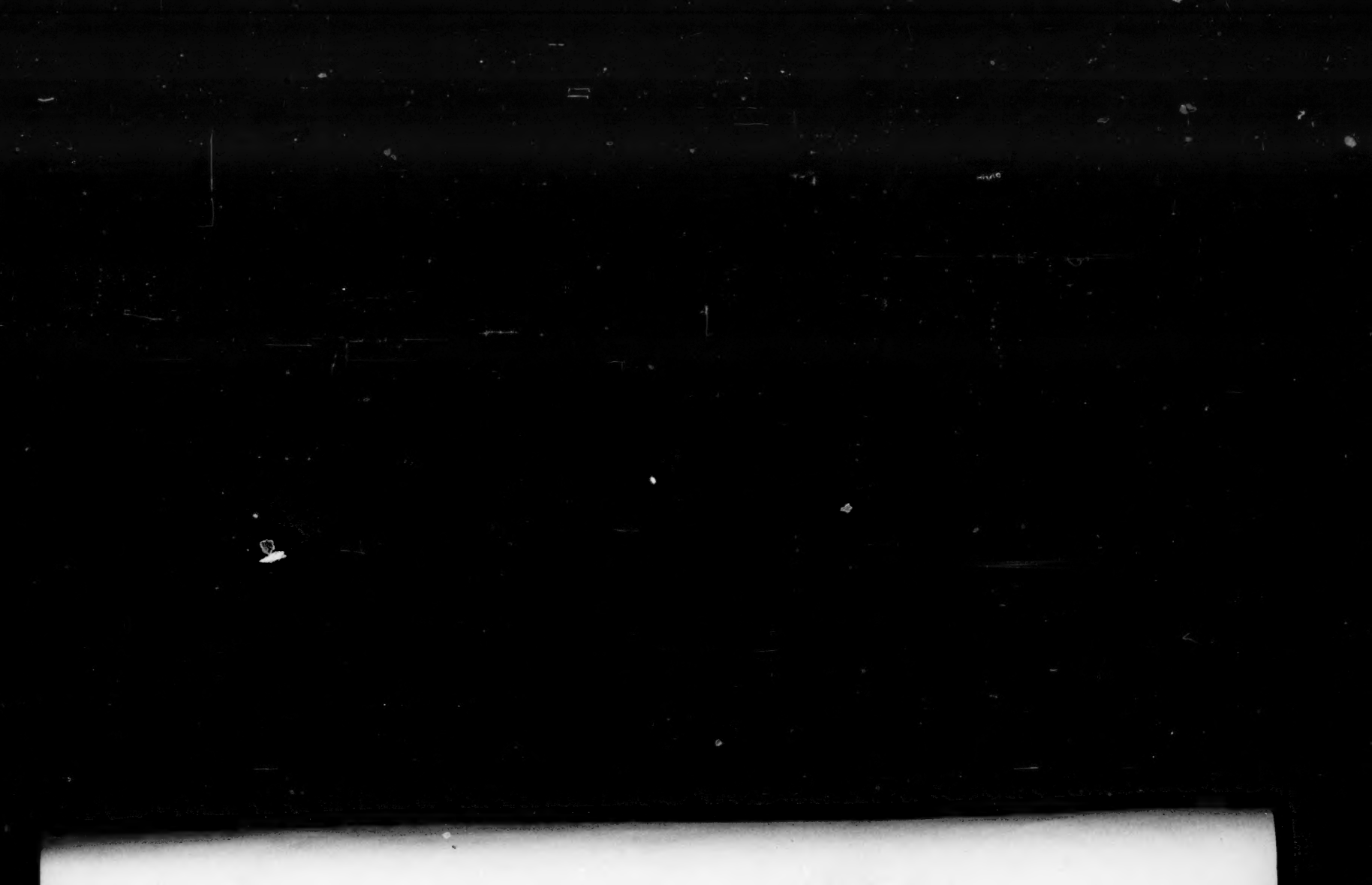
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